FINAL REPORT



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# REGULARIZATION AND APPROXIMATION OF A CLASS OF EVOLUTION PROBLEMS IN APPLIED MATHEMATICS

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Submitted by

Ralph E. Showalter
Mathematics Department

and

Graham F. Carey
Aerospace Engineering and
Engineering Mechanics Department
The University of Texas at Austin
Austin, TX 78712

FINAL REPORT

"Regularization and Approximation of a Class of Evolution Problems in Applied Mathematics"

R. Showalter and G. F. Carey

The major effort of this project has been the development of the foundations for regularization techniques related to conservation equations and some new possibly far-reaching contributions to this area. The approach that has been taken is a departure from the usual artificial viscosity type of strategies which are produced on a somewhat adhoc basis. The basic strategy is to regularize locally by a micro-structured parabolic system. A mathematical analysis of the regularized equations has been developed to support our approach. Supporting approximate analysis and numerical experiments have been made.

The development and the mathematical foundations of these microstructure models have been primary achievements of the project. The relevant nonlinear systems of partial differential equations have also been shown to provide good models of diffusion or convection of fluid or gas through a heterogeneous porous medium. Examples include flow in fissured media, problems with adsorption, heat diffusion with freezing-melting, and models for semiconductors. We have established that the problems are well-posed and developed the theory of the regularity and dependence of the solutions on data. Such information will aid approximation theory and the design of algorithms to numerically simulate solutions to these types of problems.

The major step came with the rather complete development in [1] of the linear case together with appropriate convergence and approximation results. These results were extended in part to fully nonlinear versions in [2]. Classical versions were given in [8]. This work was summarized in the review article [3] and has been received with much interest. Moreover, the applications to stationary problems [10] and to porous media [11] are underway.

We have made related numerical studies using finite elements with the regularizing strategy and the results are promising. We have also been developing some related ideas which are based on superconvergence concepts in the approximate methods used to solve both boundary and evolution PDE's [4,5]. This is a very topical research area at present as far as post processing computed solutions is concerned [13,14]. Our approach is different in that we are using the post-processing strategy to develop improved models and to develop alternative regularization strategies. This procedure also is appropriate for discrete homogenization at a macro-structure level and can be combined with statistical averaging at the micro-structure level as a regularization strategy.

Numerical studies with regularization techniques have been applied to flow calculations [6,12]. This work is being extended currently to least-squares finite element analysis including local regularization. Our previous work with a least-squares finite element formulation and parabolic regularization [7] (motivated by [9]) confirmed that this type of regularization procedure is applicable and the numerical results are positive. Some of the challenging aspects of the formulation arise in combining the microstructure regularization within the framework of a macrostructure Galerkin finite element analysis. A formulation for this embedding has been developed.

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Statement A per telecon Dr. Reza Madani
ONR/Code 1111 Arlington, VA 22217-5000

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# Diffusion of Fluid in a Fissured Medium with Micro-Structure

R.E. SHOWALTER* and N.J. WALKINGTON

Department of Mathematics
University of Texas
Austin, TX 78712

Department of Mathematics
Carnegie-Mellon University
Pittsburgh, PA 15213

## 1. Introduction

We shall study the Cauchy-Dirichlet problem for degenerate parabolic systems of the form

$$(1.1.a)\quad \frac{\partial}{\partial t}a(u) - \vec{\nabla}\cdot\widetilde{A}(x,\vec{\nabla}u) + \int_{\Gamma_x}\widetilde{B}(x,s,\vec{\nabla}_y U)\cdot\vec{\nu}\,ds \ni f\ , \qquad x\in\Omega\ ,$$

$$(1.1.b)\quad \frac{\partial}{\partial t}b(U) - \vec{\nabla}_y\cdot\widetilde{B}(x,y,\vec{\nabla}_y U) \ni F\ , \qquad y\in\Omega_x\ ,$$

$$(1.1.c)\quad \widetilde{B}(x,y,\vec{\nabla}_y U)\cdot\vec{\nu} + \mu\big(U(x,y,t) - u(x,t)\big) \ni 0\ , \qquad y\in\Gamma_x\ .$$

Here $\Omega$ is a domain in $\mathbb{R}^n$ and for each value of the macro-variable $x\in\Omega$ is specified a domain $\Omega_x$ with boundary $\Gamma_x$ for the micro-variable $y\in\Omega_x$. Each of $a, b, \mu$ is a maximal monotone graph. These graphs are not necessarily strictly increasing; they may be piecewise constant or multi-valued. The elliptic operators in (1.1.a) and (1.1.b) are nonlinear in the gradient of degree $p-1>0$ and $q-1>0$, respectively, with $\frac{1}{q}+\frac{1}{n}\geq\frac{1}{p}$, so some specific degeneracy is also permitted here. Certain first order spatial derivatives can be added to (1.1.a) and (1.1.b) with no difficulty, and corresponding problems with constraints, *i.e.*, variational inequalities, can be treated similarly. A particular example important for applications is the linear

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constraint

$$(1.1.c') \qquad U(x,y,t) = u(x,t) \ , \qquad y \in \Gamma_x \ , \ x \in \Omega$$

which then replaces (1.1.c). The system (1.1) with $\mu(s) = \frac{1}{\varepsilon}|s|^{q-2}s$ is called a *regularized micro-structure model*, and (1.1.a), (1.1.b), (1.1.c′) is the corresponding *matched micro-structure model* in which (formally) $\varepsilon \to 0$. An example of such a system as a model for the flow of a fluid (liquid or gas) through a fractured medium will be given below. In such a context, (1.1.a) prescribes the flow on the global scale of the fissure system and (1.1.b) gives the flow on the microscale of the individual cell at a specific point $x$ in the fissure system. The transfer of fluid between the cells and surrounding medium is prescribed by (1.1.c) or (1.1.c′). A major objective is to accurately model this fluid exchange between the cells and fissures.

The plan of this paper is as follows. In Section 2 we shall give the precise description and resolution of the stationary problem in a variational formulation by monotone operators from Banach spaces to their duals. In order to achieve this we describe first the relevant Sobolev spaces, the continuous direct sums of these spaces, and the distributed trace and constant functionals which occur in the system. The operators are monotone functions or multi-valued subgradients and serve as models for nonlinear elliptic equations in divergence form. We develop an abstract Green's theorem to describe the resolution of the variational form as the sum of a partial differential equation and a complementary boundary operator. Then sufficient conditions of coercivity type are given to assert the existence of generalized solutions of the variational equations. In Section 3 we describe the restriction of our system to appropriate products of $L^r$ spaces. The Hilbert space case, $r = 2$, serves not only as a convenient starting point but leads to the generalized accretive estimates we shall need for the singular case of (1.1) in which $a$ or $b$ is not only nonlinear but multi-valued. The stationary operator for (1.1) is shown to be $m$-accretive in the $L^1$ space, so we obtain a *generalized solution* in the sense of the nonlinear semigroup theory for general Banach spaces. As an intermediate step we shall show the special case of $a = b =$ identity is resolved as a *strong solution* in every $L^r$ space,

$1 < r < \infty$, and also in appropriate dual Sobolev spaces.

In order to motivate the system (1.1), let's consider the flow of a fluid through a fissured medium. This is assumed to be a structure of porous and permeable blocks or cells which are separated from each other by a highly developed system of fissures. The majority of fluid transport will occur along flow paths through the fissure system, and the relative volume of the cell structure is much larger than that of the fissure system. There is assumed to be no direct flow between adjacent cells, since they are individually isolated by the fissures, but the dynamics of the flux exchanged between each cell and its surrounding fissures is a major aspect of the model. The distributed micro-structure models that we develop here contain explicitly the local geometry of the cell matrix at each point of the fissure system, and they thereby reflect more accurately the flux exchange on the micro-scale of the individual cells across their intricate interface.

Let the flow region $\Omega$ be a bounded domain in $\mathbb{R}^n$ with boundary $\Gamma = \partial\Omega$. Let $\rho(x,t)$ and $p(x,t)$ be the *density* and *pressure*, respectively, at $x \in \Omega$ and $t > 0$, each being obtained by averaging over an appropriately small neighborhood of $x$. At each such $x$ let there be given a cell $\Omega_x$, a bounded domain in $\mathbb{R}^n$ with smooth boundary $\Gamma_x = \partial\Omega_x$. The collection of these $\Omega_x$, $x \in \Omega$, is the distribution of blocks or cells in the structure. Within each $\Omega_x$ there is fluid of density $\tilde{\rho}(x,y,t)$ and pressure $\tilde{p}(x,y,t)$, respectively, for $y \in \Omega_x$, $t > 0$. The conservation of fluid mass in the fissure system yields the global diffusion equation

$$\text{(1.2.a)} \qquad \frac{\partial}{\partial t}(\rho + a_0(p)) - \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(\rho k_j\left(\rho\frac{\partial p}{\partial x_j}\right)\frac{\partial p}{\partial x_j}\right) + q(x,t) = f(x,t)\,, \qquad x \in \Omega\,,$$

in which the total concentration $\rho + a_0(p)$ includes adsorption or capillary effects, the function $k_j$ gives the permeability of the fissure system in the $j^{th}$ coordinate direction, $q(x,t)$ is the density of mass flow of fluid into the cell $\Omega_x$ at $x$, and $f$ is the density of fluid sources. Similarly we have within each cell

$$\text{(1.2.b)} \qquad \frac{\partial}{\partial t}(\tilde{\rho} + b_0(\tilde{p})) = \sum_{j=1}^{n} \frac{\partial}{\partial y_j}\left(\tilde{\rho}\tilde{k}_j\left(\tilde{\rho}\frac{\partial \tilde{p}}{\partial y_j}\right)\frac{\partial \tilde{p}}{\partial y_j}\right)\,, \qquad y \in \Omega_x\,,$$

where $b_0$ denotes adsorption or capillary effects and the function $\tilde{k}_j$ gives the local cell permeability. Assume the flux across the cell boundary is driven by the pressure difference and is also proportional to the *average density* $\bar{\rho}$ on that pressure interval. Thus, we have the interface condition

$$\text{(1.2.c)} \qquad \sum_{j=1}^{n} \tilde{\rho}\tilde{k}_j\left(\tilde{\rho}\frac{\partial \tilde{p}}{\partial y_j}\right)\frac{\partial \tilde{p}}{\partial y_j}\nu_j + \mu\big(\bar{\rho}(\tilde{p}-p)\big) \ni 0 \,, \qquad y \in \Gamma_x \,,$$

where $\vec{\nu}$ is the unit outward normal on $\Gamma_x$ and $\mu$ is the relation between the flux across the interface and the density-weighted pressure difference as indicated. The total mass flow into the cell is given by

$$\text{(1.2.d)} \qquad q(x,t) = \int_{\Gamma_x} \tilde{\rho} \sum_{j=1}^{n} \tilde{k}_j\left(\tilde{\rho}\frac{\partial \tilde{p}}{\partial y_j}\right)\frac{\partial \tilde{p}}{\partial y_j}\nu_j \, ds \,.$$

In order to complete the dynamical system we need only to add a boundary condition on $\Gamma$ to (1.2.a) and to postulate the *state equation*

$$\text{(1.2.e)} \qquad \rho = s(p)$$

for the fluid in the fissure and cell systems. Here $s(\cdot)$ is a given monotone function (or graph) determined by the fluid.

In order to place (1.2) in a more convenient form, we introduce the monotone function

$$S(w) \equiv \int_0^w s(r)\,dr$$

and the corresponding flow potentials for the fluid in the fissures and cells

$$u = S(p) \quad , \quad U = S(\tilde{p}) \,.$$

In these variables with a change of notation the system (1.2) can be written in the form (1.1) together with boundary conditions on $\Gamma$ for $u$ or $\widetilde{A}(\nabla u)\cdot\nu$ and initial conditions at $t = 0$ on $a(u)$, $b(U)$. Note that the average density on the pressure interval $p, \tilde{p}$ is given by

$$\bar{\rho} = \frac{1}{p-\tilde{p}}\int_{\tilde{p}}^{p} s(r)\,dr = \frac{u-U}{p-\tilde{p}} \,.$$

As an alternative to (1.2.c), we could require that $\tilde{p} = p$ on $\Gamma_x$ and this leads to (1.1.c′) in place of (1.1.c). Finally, we note that the classical Forchheimer-type corrections to the Darcy Law for fluids lead to the case $p = q = 3/2$.

Systems of the form (1.1) were developed in [20], [21], [9] in physical chemistry as models for diffusion through a medium with a prescribed microstructure. Similar systems arose in soil science [4], [13] and in reservoir models for fractured media [10], [15]. By homogenization methods such systems are obtained as limits of exact micro-scale models, and then the effective coefficients are computed explicitly from local material properties [24], [16], [2]. An existence-uniqueness theory for these linear problems which exploits the strong parabolic structure of the system was given in [23]. One can alternatively eliminate $U$ and obtain a single functional differential equation for $u$ in the simpler space $L^2(\Omega)$, but the structure of the equation then obstructs the optimal parabolic type results [17]. Also see [12] for a nonlinear system with reaction-diffusion local effects.

## 2. The Variational Formulation

We begin by stating and resolving the stationary forms of our systems. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with smooth boundary, $\Gamma = \partial\Omega$. Let $1 \le p < \infty$ and denote by $L^p(\Omega)$ the space of $p^{th}$ power-integrable functions on $\Omega$, by $L^\infty(\Omega)$ the essentially bounded measurable functions, and the duality pairing by

$$(u, f)_{L(\Omega)} = \int_\Omega u(x) f(x)\, dx\,, \qquad u \in L^p(\Omega)\,,\ f \in L^{p'}(\Omega)\,,$$

for any pair of conjugate powers, $\frac{1}{p} + \frac{1}{p'} = 1$. Let $C_0^\infty(\Omega)$ denote the space of infinitely differentiable functions with compact support in $\Omega$. $W^{m,p}(\Omega)$ is the Banach space of functions in $L^p(\Omega)$ for which each partial derivative up to order $m$ belongs to $L^p(\Omega)$, and $W_0^{m,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$. See [1] for information on these Sobolev spaces. In addition, we shall be given for each $x \in \Omega$ a bounded domain $\Omega_x$ which lies locally on one side of its smooth boundary $\Gamma_x$. Let $1 < q < \infty$ and denote by $\gamma_x : W^{1,q}(\Omega_x) \to L^q(\Gamma_x)$ the *trace* map which assigns boundary values. Let $T_x$ be the range of $\gamma_x$; this is a Banach space with the norm induced by

$\gamma_x$ from $W^{1,q}(\Omega_x)$. Since $\Gamma_x$ is smooth, there is a unit outward normal $\nu_x(s)$ at each $s \in \Gamma_x$. Finally, we define $W_x^{1,q}(\Omega_x)$ to be that closed subspace consisting of those $\varphi \in W^{1,q}(\Omega_x)$ with $\gamma_x\varphi \in \mathbb{R}$, *i.e.*, each $\gamma_x(\varphi)$ is *constant* a.e. on $\Gamma_x$. We shall denote by $\vec{\nabla}_y$ the gradient on $W^{1,q}(\Omega_x)$ and by $\vec{\nabla}$ the gradient on $W^{1,p}(\Omega)$.

The essential construction to be used below is an example of a *continuous direct sum* of Banach spaces. The special case that is adequate for our purposes can be described as follows. Let $S$ be a measure space and consider the product (measure) space $Q = \Omega \times S$, where $\Omega$ has Lebesgue measure. If $U \in L^q(Q)$ then from the Fubini theorem it follows that $U(x)(z) \equiv U(x,z)$, $x \in \Omega$, $z \in S$ defines $U(x) \in L^q(S)$ at a.e. $x \in \Omega$, and for each $\Phi \in L^{q'}(Q)$

$$\int_\Omega \big(U(x), \Phi(x)\big)_{L(S)}\, dx \equiv \int_\Omega \left\{ \int_S U(x,z)\Phi(x,z)\, dz \right\} dx = \iint\limits_Q U\Phi\,.$$

Thus $L^q(Q)$ is naturally identified as a (closed) subspace of $L^q(\Omega, L^q(S))$, the Bochner $q^{th}$ integrable (equivalence classes of) functions from $\Omega$ to $L^q(S)$. Suppose $\mathrm{X} : \Omega \to \mathbb{R}$ is the characteristic function of a measurable $\Omega_* \subset \Omega$ and $w \in L^q(S)$. For each $a > 0$ we have

$$\big\{(x,z) \in Q : \mathrm{X}(x)w(z) < a\big\} = \Omega_* \times \big\{z \in S : w(z) < a\big\} \cup (\Omega \sim \Omega_*) \times S$$

and for $a \leq 0$ we delete the second term. Thus, $\mathrm{X} \cdot w$ is measurable on $Q$. It follows that each measurable step function $u = \sum \mathrm{X}_j w_j$ from $L^q(\Omega, L^q(S))$ is measurable on $Q$, and hence, belongs to $L^q(Q)$. This shows $L^q(Q)$ is dense in and therefore equal to $L^q(\Omega, L^q(S))$.

In order to prescribe a measurable family of cells, $\{\Omega_x, x \in \Omega\}$, set $S = \mathbb{R}^n$, let $Q \subseteq \Omega \times \mathbb{R}^n$ be a given measurable set, and set $\Omega_x = \{y \in \mathbb{R}^n : (x,y) \in Q\}$. Each $\Omega_x$ is measurable in $\mathbb{R}^n$ and by zero-extension we identify $L^q(Q) \hookrightarrow L^q(\Omega \times \mathbb{R}^n)$ and each $L^q(\Omega_x) \hookrightarrow L^q(\mathbb{R}^n)$. Thus we obtain from above

$$L^q(Q) \cong \left\{ U \in L^q(\Omega, L^q(\mathbb{R}^n)) : U(x) \in L^q(\Omega_x)\ ,\ \text{a.e. } x \in \Omega \right\}.$$

We shall denote the duality on this Banach space by

$$(U, \Phi)_{L(Q)} = \int_\Omega \left\{ \int_{\Omega_x} U(x,y)\Phi(x,y)\, dy \right\} dx\ , \qquad U \in L^q(Q)\ ,\ \Phi \in L^{q'}(Q)\,.$$

The *state space* for our problems will be the product $L^1(\Omega) \times L^1(Q)$.

Note that $W^{1,q}(\Omega_x)$ is continuously imbedded in $L^q(\Omega_x)$, uniformly for $x \in \Omega$. It follows that the direct sum

$$\mathcal{W}_q \equiv L^q(\Omega, W^{1,q}(\Omega_x)) \equiv \{U \in L^q(Q) : U(x) \in W^{1,q}(\Omega_x) \text{ , a.e. } x \in \Omega \text{ ,}$$

$$\text{and } \int_\Omega \|U(x)\|^q_{W^{1,q}}\, dx < \infty\}$$

is a Banach space. We shall use a variety of such spaces which can be constructed in this manner. Moreover we shall assume that each $\Omega_x$ is a bounded domain in $\mathbb{R}^n$ which lies locally on one side of its boundary, $\Gamma_x$, and $\Gamma_x$ is a $C^2$-manifold of dimension $n-1$. We assume the trace maps $\gamma_x : W^{1,q}(\Omega_x) \to L^q(\Gamma_x)$ are *uniformly* bounded. Thus for each $U \in \mathcal{W}_q$ it follows that the *distributed trace* $\gamma(U)$ defined by $\gamma(U)(x,s) \equiv (\gamma_x(U(x))(s)$, $s \in \Gamma_x$, $x \in \Omega$, belongs to $L^q(\Omega, L^q(\Gamma_x))$. The distributed trace $\gamma$ maps $\mathcal{W}_q$ onto $\mathcal{T}_q \equiv L^q(\Omega, T_x) \hookrightarrow L^q(\Omega, L^q(\Gamma_x))$.

Next consider the collection $\{W^{1,q}_x(\Omega_x) : x \in \Omega\}$ of Sobolev spaces given above and denote by $\mathcal{W}_1 \equiv L^q(\Omega, W^{1,q}_x(\Omega_x))$ the corresponding direct sum. Thus for each $U \in \mathcal{W}_1$ it follows that the distributed trace $\gamma(U)$ belongs to $L^q(\Omega)$. We define $\mathcal{W}^{1,p}_0$ to be the subspace of those $U \in \mathcal{W}_1$ for which $\gamma(U) \in W^{1,p}_0(\Omega)$. Since $\gamma : \mathcal{W}_1 \to L^q(\Omega)$ is continuous, $\mathcal{W}^{1,p}_0$ is complete with the norm

$$\|U\|_{\mathcal{W}^{1,p}_0} \equiv \|U\|_{\mathcal{W}_q} + \|\gamma U\|_{W^{1,p}_0} \text{ .}$$

This Banach space $W^{1,p}_0(\Omega) \times \mathcal{W}_q$ will be the *energy space* for the regularized problem (1.1) and $\mathcal{W}^{1,p}_0$ will be the energy space for the constrained problem in which (1.1.c) is replaced by the Dirichlet condition (1.1.c′). Note that $\mathcal{W}^{1,p}_0$ is identified with the closed subspace $\{[\gamma U, U] : U \in \mathcal{W}^{1,p}_0\}$ of $W^{1,p}_0(\Omega) \times \mathcal{W}_q$. Finally, we shall let $\mathcal{W}_0$ denote the kernel of $\gamma$, $\mathcal{W}_0 = \{U \in \mathcal{W}_q : \gamma U = 0 \text{ in } \mathcal{T}_q\}$.

We have defined $W^{1,q}_x(\Omega_x)$ to be the set of $w \in W^{1,q}(\Omega_x)$ for which $\gamma_x w$ is a constant multiple of $\mathbf{1}_x$, the constant function equal to one on $\Gamma_x$. Thus $W^{1,q}_x(\Omega_x)$ is the pre-image by $\gamma_x$ of the subspace $\mathbb{R}\cdot\mathbf{1}_x$ of $T_x$. We specified the subspace $\mathcal{W}_1$ similarly as the subspace of $\mathcal{W}_q$ obtained as the pre-image by $\gamma$ of the subspace

$L^q(\Omega)$ of $\mathcal{T}_q$. To be precise, we denote by $\lambda$ the map of $L^q(\Omega)$ into $\mathcal{T}_q$ given by $\lambda v(x) = v(x) \cdot \mathbf{1}_x$, a.e. $x \in \Omega$, $v \in L^q(\Omega)$; $\lambda$ is an isomorphism of $L^q(\Omega)$ onto a closed subspace of $\mathcal{T}_q$. The dual map $\lambda'$ taking $\mathcal{T}_q'$ into $L^{q'}(\Omega)$ is given by

$$\lambda' g(v) = g(\lambda v) = \int_\Omega g_x(\mathbf{1}_x) \cdot v(x)\, dx \ , \qquad g \in \mathcal{T}_q' \ , \ v \in L^q(\Omega) \ ,$$

so we have $\lambda' g(x) = g_x(\mathbf{1}_x)$, a.e. $x \in \Omega$. Moreover, when $g_x \in L^{q'}(\Gamma_x)$ it follows that

$$g_x(\mathbf{1}_x) = \int_{\Gamma_x} g_x(y)\, dy \ ,$$

the integral of the indicated boundary functional. Thus, for $g \in L^{q'}(\Omega, L^{q'}(\Gamma_x)) \subset \mathcal{T}_q'$, $\lambda' g \in L^{q'}(\Omega)$ is given by

$$\lambda' g(x) = \int_{\Gamma_x} g_x(y)\, dy \ , \quad \text{a.e. } x \in \Omega \ . \tag{2.1}$$

The imbedding $\lambda$ of $L^q(\Omega)$ into $\mathcal{T}_q$ and its dual map $\lambda$ will play an essential role in our system below.

We consider elliptic differential operators in divergence form as realizations of monotone operators from Banach spaces to their duals. Assume we are given $\widetilde{A} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ such that for some $1 < p < \infty$, $g_1 \in L^{p'}(\Omega)$, $g_0 \in L^1(\Omega)$, $c$ and $c_0 > 0$

(2.2.a) $\widetilde{A}(x, \vec{\xi})$ is continuous in $\vec{\xi} \in \mathbb{R}^n$ and measurable in $x$, and

$$|\widetilde{A}(x, \vec{\xi})| \leq c|\xi|^{p-1} + g_1(x) \ ,$$

(2.2.b) $\langle \widetilde{A}(x, \vec{\xi}) - \widetilde{A}(x, \vec{\eta}), \vec{\xi} - \vec{\eta} \rangle \geq 0$ ,

(2.2.c) $\widetilde{A}(x, \vec{\xi}) \cdot \vec{\xi} \geq c_0 |\vec{\xi}|^p - g_0(x)$

for a.e. $x \in \Omega$ and all $\vec{\xi}, \vec{\eta} \in \mathbb{R}^n$.

Then the global diffusion operator $\mathcal{A} : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is given by

$$\mathcal{A}u(v) = \int_\Omega \widetilde{A}(x, \vec{\nabla} u(x)) \vec{\nabla} v(x)\, dx \ , \qquad u, v \in W_0^{1,p}(\Omega) \ .$$

Thus, each $\mathcal{A}u$ is equivalent to its restriction to $C_0^\infty(\Omega)$, the distribution

$$Au \equiv \mathcal{A}u\big|_{C_0^\infty(\Omega)} = -\vec{\nabla}\cdot\widetilde{A}(\cdot,\vec{\nabla}u)$$

which specifies the value of this nonlinear elliptic divergence operator.

In order to specify a collection of local diffusion operators, $\mathcal{B}_x : W^{1,q}(\Omega_x) \to W^{1,q}(\Omega_x)'$, assume we are given $\widetilde{B} : Q \times \mathbb{R}^n \to \mathbb{R}^n$ such that for some $1 < q < \infty$, $h_1 \in L^{q'}(Q)$, $h_0 \in L^1(Q)$, $c$ and $c_0 > 0$

(2.3.a) $\widetilde{B}(x,y,\vec{\xi})$ is continuous in $\vec{\xi} \in \mathbb{R}^n$ and measurable in $(x,y) \in Q$, and

$$|\widetilde{B}(x,y,\vec{\xi})| \le c|\vec{\xi}|^{q-1} + h_1(x,y) \ ,$$

(2.3.b) $\langle \widetilde{B}(x,y,\vec{\xi}) - \widetilde{B}(x,y,\vec{\eta}),\ \vec{\xi}-\vec{\eta}\rangle \ge 0$ ,

(2.3.c) $\widetilde{B}(x,y,\vec{\xi})\cdot\vec{\xi} \ge c_0|\vec{\xi}|^q - h_0(x,y)$

for a.e. $(x,y) \in Q$ and all $\vec{\xi},\vec{\eta} \in \mathbb{R}^n$.

Then define for each $x \in \Omega$

$$\mathcal{B}_x w(v) = \int_{\Omega_x} \widetilde{B}\big(x,y,\vec{\nabla}_y w(y)\big)\vec{\nabla}_y v(y)\,dy \ , \qquad w,v \in W^{1,q}(\Omega_x) \ .$$

The elliptic differential operator on $\Omega_x$ is given by the *formal part* of $\mathcal{B}_x$, the distribution

$$B_x w \equiv \mathcal{B}_x w\big|_{C_0^\infty(\Omega_x)} = -\vec{\nabla}_y\cdot\widetilde{B}(x,\cdot,\vec{\nabla}_y w)$$

in $W_0^{1,q}(\Omega_x)'$. Also, we shall denote by $\mathcal{B} : \mathcal{W}_q \to \mathcal{W}_q'$ the *distributed* operator constructed from the collection $\{\mathcal{B}_x : x \in \Omega\}$ by

$$\mathcal{B}U(x) = \mathcal{B}_x\big(U(x)\big) \ , \qquad \text{a.e. } \ x \in \Omega \ , \ U \in \mathcal{W}_q \ ,$$

and we note that this is equivalent to

$$\mathcal{B}U(V) \equiv \int_\Omega \mathcal{B}_x\big(U(x)\big)V(x)\,dx \ , \qquad U,V \in \mathcal{W}_q \ .$$

The coupling term in our system will be given as a monotone graph which is a subgradient operator. Thus, assume $m : \mathbb{R} \to \mathbb{R}^+$ is convex and bounded by

$$m(s) \le C(|s|^q + 1) \ , \qquad s \in \mathbb{R} \ , \tag{2.4}$$

hence, continuous. Then by

$$\tilde{m}(g) \equiv \int_\Omega \int_{\Gamma_x} m(g(x,s))\, ds\, dx\ , \qquad g \in L^q(\Omega, L^q(\Gamma_x))$$

we obtain the convex, continuous $\tilde{m} : L^q(\Omega, L^q(\Gamma_x)) \to \mathbb{R}^+$. Assume $\frac{1}{q} + \frac{1}{n} \geq \frac{1}{p}$ so that $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, and consider the linear continuous maps

$$\lambda : W_0^{1,p}(\Omega) \to L^q(\Omega, L^q(\Gamma_x)) \quad , \quad \gamma : \mathcal{W}_q \to L^q(\Omega, L^q(\Gamma_x))\ .$$

Then the composite function

$$M[u,U] \equiv \tilde{m}(\gamma U - \lambda u)\ , \qquad u \in W_0^{1,p}(\Omega)\ ,\ U \in \mathcal{W}_q\ ,$$

is convex and continuous on $W_0^{1,p}(\Omega) \times \mathcal{W}_q$. The subgradients are directly computed by standard results [11]. Specifically, we have $\hat{g} \in \partial\tilde{m}(g)$ if and only if

$$\hat{g}(x,s) \in \partial m(g(x,s))\ , \quad \text{a.e. } s \in \Gamma_x\ , \quad \text{a.e. } x \in \Omega\ ,$$

and we have $[f,F] \in \partial M[u,U]$ if and only if $f = -\lambda'(\mu)$ in $W^{-1,p'}(\Omega)$ and $F = \gamma'(\mu)$ in $\mathcal{W}_q'$ for some $\mu \in \partial\tilde{m}(\gamma U - \lambda u)$.

The following result gives sufficient conditions for the *stationary regularized problem* to be well-posed.

**Proposition 1.** *Assume $1 < p, q$, $\frac{1}{q} + \frac{1}{n} \geq \frac{1}{p}$, and define the spaces and operators $\lambda, \gamma$ as above. Specifically, the sets $\{\Omega_x : x \in \Omega\}$ are uniformly bounded with smooth boundaries, and the trace maps $\{\gamma_x\}$ are uniformly bounded. Let the functions $\widetilde{A}, \widetilde{B}$, and $m$ satisfy (2.2), (2.3) (2.4), and assume in addition that*

$$m(s) \geq c_0 |s|^q\ , \qquad s \in \mathbb{R}\ . \tag{2.5}$$

*Then for each pair $f \in W^{-1,p'}(\Omega)$, $F \in \mathcal{W}_q'$ there exists a solution of*

$$u \in W_0^{1,p}(\Omega) : A(u) - \lambda'(\mu) = f \quad \text{in} \quad W^{-1,p'}(\Omega) \tag{2.6.a}$$

$$U \in \mathcal{W}_q : \mathcal{B}(U) + \gamma'(\mu) = F \quad \text{in } \mathcal{W}_q' \tag{2.6.b}$$

$$\mu \in L^{q'}(\Omega, L^{q'}(\Gamma_x)) : \mu \in \partial\tilde{m}(\gamma U - \lambda u)\ . \tag{2.6.c}$$

*For any such solution we have*

$$(2.7) \qquad \int_{\Gamma_x} \mu(x,s)\,ds = \langle F(x), 1_x \rangle \,, \qquad \text{a.e. } x \in \Omega \,,$$

*where $1_x$ denotes the constant unit function in $W^{1,q}(\Omega_x)$.*

*Proof.* The system (2.6) is a "pseudo-monotone plus subgradient" operator equation of the form

$$(2.6') \qquad \begin{aligned} &[u,U] \in W_0^{1,p}(\Omega) \times \mathcal{W}_q : \quad \text{for all } [v,V] \in W_0^{1,p}(\Omega) \times \mathcal{W}_q \\ &\mathcal{A}u(v) + \mathcal{B}U(V) + \partial M[u,U]([v,V]) \ni f(v) + F(V) \,. \end{aligned}$$

It remains only to verify a coercivity condition, namely,

$$(2.8) \qquad \frac{\mathcal{A}u(u) + \mathcal{B}U(U) + \tilde{m}(\gamma U - \lambda u)}{\|u\|_{W_0^{1,p}(\Omega)} + \|U\|_{\mathcal{W}_q}} \to +\infty$$

$$\text{as } \|u\|_{W_0^{1,p}(\Omega)} + \|U\|_{\mathcal{W}_q} \to +\infty \,.$$

Choose $k = \max\{|y_n| : y \in \Omega_x, x \in \Omega\}$ and let $\nu_x = (\nu_x^1, \ldots, \nu_x^n)$ be the unit normal on $\Gamma_x$. For $v \in W^{1,q}(\Omega_x)$ we have by Gauss' Theorem

$$\begin{aligned} \int_{\Omega_x} \left(|v|^q + y_n q |v|^{q-1} \partial_n v\right) &= \int_{\Omega_x} \partial_n \left(y_n |v(y)|^q\right) dy \\ &= \int_{\Gamma_x} \nu_x^n(s) s_n |\gamma_x v(s)|^q \, ds \,. \end{aligned}$$

Hölder's inequality then shows

$$\|v\|_{L^q(\Omega_x)}^q \le k \|\gamma_x v\|_{L^q(\Gamma_x)}^q + qk \|v\|_{L^q(\Omega_x)}^{q-1} \|\partial_n v\|_{L^q(\Omega_x)} \,,$$

and from this follows

$$\|v\|_{L^q(\Omega_x)}^q \le 2k \|\gamma_x v\|_{L^q(\Gamma_x)}^q + (2k)^q (q-1)^{q-1} \|\partial_n v\|_{L^q(\Omega_x)}^q$$

by Young's inequality. From here we obtain

$$(2.9) \qquad c_0 \|V\|_{L^q(Q)}^q \le \|\gamma V\|_{L^q(\Omega, L^q(\Gamma_x))}^q + \|\nabla_y V\|_{L^q(Q)}^q \,, \quad V \in \mathcal{W}_q \,.$$

Thus from the a-priori estimate

$$\begin{aligned}
&\mathcal{A}u(u) + \mathcal{B}U(U) + M(\gamma U - \lambda u) \geq \\
(2.10) \qquad & c_0\|\vec{\nabla} u\|^p_{L^p(\Omega)} - \|g_0\|_{L^1(\Omega)} + c_0\|\vec{\nabla}_y U\|^q_{L^q(Q)} - \|h_0\|_{L^1(Q)} \\
&\quad + c_0\|\gamma U - \lambda u\|^q_{L^q(\Omega, L^q(\Gamma_x))} , \qquad u \in W_0^{1,p}(\Omega) , \ U \in \mathcal{W}_q ,
\end{aligned}$$

the Poincaré-type inequality (2.9) and the equivalence of $\|\nabla u\|_{L^p(\Omega)}$ with the norm on $W_0^{1,p}(\Omega)$, we can obtain the coercivity condition (2.8). Specifically, if (2.8) is bounded by $K$, then (2.10) is bounded above by

$$\begin{aligned}
&K\big(\|u\|_{W_0^{1,p}(\Omega)} + \|\nabla_y U\|_{L^q(Q)} + \|\gamma U\|_{L^q(\Omega, L^q(\Gamma_x))}\big) \\
&\quad \leq K\big(\|u\|_{W_0^{1,p}(\Omega)} + \|\nabla_y U\|_{L^q(Q)} + \|\gamma U - \lambda\varphi\|_{L^q(\Omega, L^q(\Gamma_x))} + \|\lambda u\|_{L^q(\Omega)}\big) ,
\end{aligned}$$

and the last term is dominated by the first. This gives an explicit bound on each of these terms and, hence, on $\|u\|_{W_0^{1,p}(\Omega)} + \|U\|_{\mathcal{W}_q}$.

Finally, we apply (2.6.b) to the function $V \in \mathcal{W}_q$ given by $V(x, y) = v(x)$ for some $v \in L^q(\Omega)$, and this shows

$$\mu(\gamma v) = \langle F, v \rangle$$

since $\mathcal{B}U(V) = 0$, and thus

$$\int_\Omega \lambda' \mu(x) v(x)\, dx = \mu(\lambda v) = \mu(\gamma v) = \int_\Omega \langle F(x), 1 \rangle v(x)\, dx .$$

The identity (2.7) now follows from (2.1).

For the more general case of the *degenerate stationary problem* corresponding to (1.1), we obtain the following result.

**Corollary 1.** *Let $\varphi : \mathbb{R} \to \mathbb{R}^+$ and $\Phi : \mathbb{R} \to \mathbb{R}^+$ be convex and continuous, with $\varphi(0) = \Phi(0) = 0$, and assume*

$$(2.11) \qquad \varphi(s) \leq C\big(|s|^q + 1\big) \quad , \quad \Phi(s) \leq C\big(|s|^q + 1\big) , \qquad s \in \mathbb{R} .$$

*For each pair $f \in W^{-1,p'}(\Omega)$, $F \in \mathcal{W}_q'$, there exists a solution of*

(2.12.a) $$u \in W_0^{1,p}(\Omega) : a + A(u) - \lambda'(\mu) = f \quad in \quad W^{-1,p'}(\Omega)$$

(2.12.b) $$U \in \mathcal{W}_q : b + \mathcal{B}(U) + \gamma'(\mu) = F \quad in \quad \mathcal{W}_q'$$

(2.12.c) $$\mu \in \partial\tilde{m}(\gamma U - \lambda u) \quad in \quad L^{q'}(\Omega, L^{q'}(\Gamma_x)) \,, \quad and$$

(2.12.d) $$a \in \partial\varphi(u) \quad in \quad L^{q'}(\Omega) \,, \; b \in \partial\Phi(U) \quad in \quad L^{q'}(Q) \,.$$

*For any such solution we have*

(2.13) $$\int_{\Omega_x} b(x,y)\, dy + \int_{\Gamma_x} \mu(x,s)\, ds = \langle F(x), 1_x \rangle \,, \quad \text{a.e. } x \in \Omega \,.$$

*Proof.* This follows as above but with the continuous convex function

$$\Psi[u,U] = \int_\Omega \varphi\big(u(x)\big)\, dx + \int_\Omega \int_{\Omega_x} \Phi\big(U(x,y)\big)\, dy\, dx + \tilde{m}(\gamma U - \lambda u) \,, \; [u,U] \in W_0^{1,p}(\Omega) \times \mathcal{W}_q \,.$$

The subgradient can be computed termwise because the three terms are continuous on $L^q(\Omega)$, $L^q(Q)$, and $L^q(\Omega, L^q(\Gamma_x))$, respectively.

*Remark.* The lower bound (2.5) on $m(\cdot)$ may be deleted in Corollary 1 if such a lower estimate is known to hold for $\Phi$. It is also unnecessary in the matched microstructure model; see below.

In order to prescribe the boundary condition (1.1.c) explicitly, we develop an appropriate Green's formula for the operators $B_x$. Note that we can identify $L^{q'}(\Omega_x) \subset W^{-1,q'}(\Omega_x)$ since $W_0^{1,q}(\Omega_x)$ is dense in $L^q(\Omega_x)$, so it is meaningful to define

$$D_x \equiv \{w \in W^{1,q}(\Omega_x) : B_x w \in L^{q'}(\Omega_x)\} \,.$$

This is the domain for the *abstract Green's Theorem*.

Lemma 1. *There is a unique operator $\partial_x : D_x \to T_x'$ for which $\mathcal{B}_x w = B_x w + \gamma_x' \partial_x w$ for all $w \in D_x$. That is, we have*

(2.14) $$\mathcal{B}_x w(v) = (B_x w, v)_{L(\Omega_x)} + \langle \partial_x w, \gamma_x v \rangle \,, \qquad v \in W^{1,q}(\Omega_x) \,,$$

for every $w \in D_x$.

*Proof.* The strict morphism $\gamma_x$ of $W^{1,q}(\Omega_x)$ onto $T_x$ has a dual $\gamma'_x$ which is an isomorphism of $T'_x$ onto $W_0^{1,q}(\Omega_x)^\perp$, the annihilator in $W^{1,q}(\Omega_x)'$ of the kernel of $\gamma_x$. For each $w \in D_x$, the difference $\mathcal{B}_x w - B_x w$ is in $W_0^{1,q}(\Omega_x)^\perp$, so it is equal to $\gamma'_x(\partial_x w)$ for a unique element $\partial_x w \in T'_x$.

*Remark.* The identity (2.14) is a generalized decomposition of $\mathcal{B}_x$ into a partial differential operator on $\Omega_x$ and a boundary condition on $\Gamma_x$. If $\Gamma_x$ is smooth, $\nu_x$ denotes the unit outward normal on $\Gamma_x$, and if $\tilde{B}(x,\cdot,\vec{\nabla}_y w) \in [W^{1,q'}(\Omega_x)]^n$, then $w \in D_x$ and from the classical Green's Theorem we obtain

$$\mathcal{B}_x w(v) - (B_x w, v)_{L(\Omega_x)} = \int_{\Gamma_x} \tilde{B}(x,s,\vec{\nabla}_y w)\vec{\nu}_x(s)\gamma v(s)\,ds\,, \qquad v \in W^{1,q}(\Omega_x)\,.$$

Thus, $\partial_x w = \tilde{B}(x,\cdot,\vec{\nabla}_y w)\cdot\vec{\nu}_x$ is the indicated normal derivative in $L^{q'}(\Gamma_x)$ when $\tilde{B}(x,\cdot,\vec{\nabla}_y w)$ is as smooth as above, and so we can regard $\partial_x w$ in general as an extension of this nonlinear differential operator on the boundary.

The *formal part* of $\mathcal{B} : \mathcal{W}_q \to \mathcal{W}'_q$ is the operator $B : \mathcal{W}_q \to \mathcal{W}'_0$ given by the restriction $B(U) \equiv \mathcal{B}U|_{\mathcal{W}_0}$. Since $\mathcal{W}_0$ is dense in $L^q(Q)$ we can specify the domain

$$D \equiv \{U \in \mathcal{W}_q : B(U) \in L^{q'}(Q)\}$$

on which we obtain as before a distributed form of Green's theorem.

**Lemma 2.** *There is a unique operator $\partial : D \to T'_q$ such that*

$$\mathcal{B}(U)(V) = (B(U),V)_{L(Q)} + \langle \partial U, \gamma V\rangle\,, \qquad U \in D\,,\ V \in \mathcal{W}_q\,.$$

**Proposition 2.** *Let the Sobolev spaces and trace operators be given as above. We* *summarize them in the following diagrams*

$$
\begin{array}{ccccccc}
L^q(\Omega_x) & & L^q(\Gamma_x) & \qquad & L^q(Q) & & L^q(\Omega, L^q(\Gamma_x)) \\
\cup & & \cup & & \cup & & \cup \\
W^{1,q}(\Omega_x) & \xrightarrow{\gamma_x} & T_x & & \mathcal{W}_q & \xrightarrow{\gamma} & \mathcal{T}_q \\
\cup & & \lambda_x \hookuparrow & & \cup & & \lambda \hookuparrow \\
W_x^{1,q}(\Omega_x) & \longrightarrow & \mathbb{R}\cdot 1_x & & \mathcal{W}_1 & \xrightarrow{\gamma_1} & L^q(\Omega) \\
\cup & & \cup & & \cup & & \uparrow \\
W_0^{1,q}(\Omega_x) & \longrightarrow & \{0\} & & \mathcal{W}_0 & \longrightarrow & \{0\}
\end{array}
$$

*in which $\gamma_1$ is the restriction of $\gamma$ to $\mathcal{W}_1$. $W_0^{1,q}(\Omega_x)$, $\mathcal{W}_0$ are dense in $L^q(\Omega_x)$, $L^q(Q)$, respectively. Let operator $\mathcal{B}_x$, $x \in \Omega$, and $\mathcal{B}$ be given and define their formal parts $B_x$, $B$ as above. Then construct the domains $D_x$, $D$ and boundary operators $\partial_x$, $\partial$ as in Lemma 1 and Lemma 2, respectively. It follows that for any $U \in \mathcal{W}_q$,*

*(a) $BU(x) = B_x(U(x))$ in $W_0^{1,q}(\Omega_x)'$ for a.e. $x \in \Omega$, and $U \in D$ if and only if $U(x) \in D_x$ for a.e. $x \in \Omega$ and $x \mapsto B_xU(x)$ belongs to $L^{q'}(Q)$;*

*(b) for each $U \in D$,*

$$\partial U(x) = \partial_x\big(U(x)\big) \quad \text{in } T_x' \text{ for a.e. } x \in \Omega$$

*and*

$$\mathcal{B}U = BU + \gamma_1'(\lambda'\partial U) \quad \text{in } \mathcal{W}_1' \ ,$$

*and for each $V \in \mathcal{W}_1$ we have*

$$\int_\Omega \mathcal{B}_xU(x)\big(V(x)\big)\,dx = \int_Q B_xU(x)V(x)\,dy\,dx + \int_\Omega \langle \partial_xU(x), 1_x\rangle(\gamma_1 V)(x)\,dx \ .$$

*Proof.* (a) For $V \in \mathcal{W}_0$ we obtain from the definitions of $B$, $\mathcal{B}$ and $B_x$, respectively,

$$\int_\Omega BU(x)V(x)\,dx = \int_\Omega \mathcal{B}U(V)\,dx = \int_\Omega \mathcal{B}_xU(x)\big(V(x)\big)\,dx = \int_\Omega B_xU(x)V(x)\,dx \ ,$$

and so the first equality holds since $\mathcal{W}_0' = L^{q'}(\Omega, W_0^{1,q}(\Omega_x)')$. The characterization of $D$ is immediate now.

(b) For $V \in \mathcal{W}_q$ we obtain from the definitions of $\gamma, \partial, \partial_x$, respectively, and (a)

$$\int_\Omega \partial U\big(\gamma_x V(x)\big)\,dx = \int_\Omega \partial U(\gamma V)\,dx = \int_\Omega (\mathcal{B}U - BU)(x)V(x)\,dx$$

$$= \int_\Omega \partial_x\big(U(x)\big)\gamma_x V(x)\,dx\,.$$

Since the range of $\gamma$ is $T_q' = L^{q'}(\Omega, T_x')$, the first equality follows. The second is immediate from Lemma 2 since on $\mathcal{W}_1$, $\gamma = \lambda \circ \gamma_1$ and $\gamma' = \gamma_1' \lambda'$, and the third follows from the preceding remarks.

**Corollary 2.** *In the situation of Corollary 1, $f \in L^{q'}(\Omega)$ and $F \in L^{q'}(Q)$ if and only if $Au \in L^{q'}(\Omega)$ and $B(U) \in L^{q'}(Q)$, and in that case the solution satisfies almost everywhere*

$$a(x) \in \partial\varphi\big(u(x)\big)\ ,\ a(x) + Au(x) + \int_{\Omega_x} b(x,y)\,dy = f(x) + \int_{\Omega_x} F(x,y)\,dy\ ,\ x \in \Omega\ ,$$

$$u(s) = 0\ , \qquad s \in \Gamma\ ,$$

$$b(x,y) \in \partial\Phi\big(U(x,y)\big)\ , \quad b(x,y) + BU(x,y) = F(x,y)\ , \qquad y \in \Omega_x$$

$$\mu(x,s) \in \partial m\big(\gamma U(x,s) - u(x)\big)\ , \quad \partial_x\big(U(x)\big)(s) + \mu(x,s) = 0\ , \qquad s \in \Gamma_x\,.$$

Finally, we note that corresponding results for the ***stationary matched microstructure model*** are obtained directly by specializing the system (2.6′) to the space $\mathcal{W}_0^{1,p}$. This is identified with $\{[\gamma U, U] : U \in \mathcal{W}_0^{1,p}\}$ as a subspace of $W_0^{1,p}(\Omega) \times \mathcal{W}_q$, and we need only to restrict the solution $[u, U]$ and the test functions $[v, V]$, $v = \gamma V$, to this subspace to resolve the matched model. Then the coupling term $\mathcal{M}$ does not occur in the system; see the proof of Proposition 1, especially for the coercivity. These observations yield the following analogous results for the matched microstructure model.

**Proposition 1′.** *Assume $1 < p, q, \frac{1}{q} + \frac{1}{n} \geq \frac{1}{p}$, and define the spaces and operators $\lambda, \gamma$ as before. Let the functions $\widetilde{A}, \widetilde{B}$, and $m$ satisfy (2.2), (2.3) and (2.4). Then for each pair $f \in W^{-1,p'}(\Omega)$, $F \in \mathcal{W}_1'$ there exists a unique solution of*

(2.15.a) $$u \in W_0^{1,p}(\Omega) : A(u) = f + \langle F, 1 \rangle \ \ in \ \ W^{-1,p'}(\Omega)$$

(2.15.b) $\quad U \in \mathcal{W}_1 : B(U) = F \ \text{ in } \ \mathcal{W}_0'$

(2.15.c) $\quad \gamma U = \lambda u \ \text{ in } \ L^q(\Omega) \subset \mathcal{T}_q$ .

**Corollary 1′.** *Suppose $\varphi, \Phi$ are given as before and assume (2.11). For $f, F$ as above there exists a unique solution of*

(2.16.a) $\quad u \in W_0^{1,p}(\Omega) : a + \langle b, 1\rangle + A(u) = f + \langle F, 1\rangle \ \text{ in } \ W^{-1,p'}(\Omega)$

(2.16.b) $\quad U \in \mathcal{W}_1 : b + B(U) = F \ \text{ in } \ \mathcal{W}_0'$

(2.16.c) $\quad \gamma U = \lambda u \ \text{ in } \ L^q(\Omega) \subset \mathcal{T}_q$

(2.16.d) $\quad a \in \partial\varphi(u) \ \text{ in } \ L^{q'}(\Omega) \ , \ b \in \partial\Phi(U) \ \text{ in } \ L^{q'}(Q)$ .

*In addition, $f \in L^{q'}(\Omega)$ and $F \in L^{q'}(Q)$ if and only if $Au \in L^{q'}(\Omega)$ and $B(U) \in L^{q'}(Q)$, and in that case the solution satisfies almost everywhere*

$$
\begin{aligned}
&a(x) \in \partial\varphi(u(x)) \ , \ a(x) + Au(x) + \int_{\Omega_x} b(x,y)\,dy = f(x) + \int_{\Omega_x} F(x,y)\,dy \ , && x \in \Omega \ , \\
&u(s) = 0 && s \in \Gamma \ , \\
&b(x,y) \in \partial\Phi(U(x,y)) \ , \ b(x,y) + BU(x,y) = F(x,y) \ , && y \in \Omega_x \ , \\
&U(x,s) = u(x) \ , && s \in \Gamma_x \ .
\end{aligned}
$$

*Remark.* For the very special case of $p = q \geq 2$ and $a(u) = u$, $b(U) = U$ in the situation of Proposition 1 it follows from [6] or [19] that the Cauchy-Dirichlet problem for (1.1) is well-posed in the space $L^p(0,T; W_0^{1,p}(\Omega) \times \mathcal{W}_p)$ with appropriate initial data $u(x,0)$, $U(x,y,0)$ and source functions $f(x,t)$, $F(x,y,t)$. A similar remark holds in the case of Proposition 1′ for the matched model with (1.1.c′). These restrictive assumptions will be substantially relaxed in the next section.

Furthermore, variational inequalities may be resolved for problems corresponding to either the regularized or the matched microstructure model by adding the indicator function of a convex constraint set to the convex function $\Psi$. Thus one can handle such problems with constraints on the global variable $u$, the local variables $U$, or their difference $\lambda u - \gamma U$ on the interface.

## 3. The $L^r$-Operators

Assume we are in the situation of Proposition 1. We define a relation or multi-valued operator $\mathrm{C}_2$ on the Hilbert space $L^2(\Omega) \times L^2(Q)$ as follows: $\mathrm{C}_2[u, U] \ni [f, F]$ if and only if

(3.1.a) $$u \in L^2(\Omega) \cap W_0^{1,p}(\Omega) : \mathcal{A}(u) - \lambda' \mu = f \in L^2(\Omega)$$

(3.1.b) $$U \in L^2(Q) \cap \mathcal{W}_q : \mathcal{B}(U) + \gamma' \mu = F \in L^2(Q)$$

$$\text{for some } \mu \in \partial \tilde{m}(\gamma U - \lambda u) \text{ in } L^{q'}(\Omega, L^{q'}(\Gamma_x)).$$

Thus, $\mathrm{C}_2$ is the restriction of (2.6) to $L^2(\Omega) \times L^2(Q)$. Note that $\lambda' \mu \in L^2(\Omega)$ by (2.7).

**Lemma 3.** *If $\sigma : \mathbb{R} \to \mathbb{R}$ is monotone, Lipschitz, and $\sigma(0) = 0$, then for each pair*

$$\mathrm{C}_2[u_j, U_j] \ni [f_j, F_j], \qquad j = 1, 2,$$

*there follows*

$$\big(f_1 - f_2, \sigma(u_1 - u_2)\big)_{L^2(\Omega)} + \big(F_1 - F_2, \sigma(U_1 - U_2)\big)_{L^2(Q)} \geq 0 \,.$$

*Proof.* Since $\sigma$ is Lipschitz and $\sigma(0) = 0$, we have $\sigma(u_1 - u_2) \in W_0^{1,p}(\Omega)$ and $\sigma(U_1 - U_2) \in \mathcal{W}_q$. Also the chain rule applies to these functions, so we compute

$$\langle \mathcal{A}u_1 - \mathcal{A}u_2, \sigma(u_1, u_2) \rangle = \int_\Omega (\tilde{A}(x, \vec{\nabla} u_1) - \tilde{A}(x, \nabla u_2)) \vec{\nabla}(u_1 - u_2) \sigma'(u_1 - u_2) \, dx \,,$$

$$\langle \mathcal{B}U_1 - \mathcal{B}U_2, \sigma(U_1 - U_2) \rangle =$$

$$\int_\Omega \int_{\Omega_x} (\tilde{B}(x, y, \vec{\nabla}_y U_1) - \tilde{B}(x, y, \vec{\nabla}_y U_2)) \vec{\nabla}_y (U_1 - U_2) \sigma'(U_1 - U_2) \, dy \, dx \,.$$

Both of these are non-negative because of (2.1.b), (2.2.b) and $\sigma' \geq 0$. The remaining term to check is

$$\left(-\lambda'(\mu_1 - \mu_2),\, \sigma(u_1 - u_2)\right)_{L^2(\Omega)} + \langle \gamma'(\mu_1 - \mu_2),\, \sigma(U_1 - U_2)\rangle$$

$$= \int_\Omega \int_{\Gamma_x} (\mu_1(x,s) - \mu_2(x,s))\left(\sigma(\gamma U_1 - \gamma U_2) - \sigma(\lambda u_1 - \lambda u_2)\right) ds\, dx\ .$$

Since $\sigma$ is a monotone function and $\partial m$ is a monotone graph, this integrand is non-negative and the result follows.

As a consequence of Lemma 3 with $\sigma(s) = s$, the operator $C_2$ is monotone on the Hilbert space $L^2(\Omega) \times L^2(Q)$. Moreover, we obtain the following.

**Proposition 3.** *The operator $C_2$ is maximal monotone on $L^2(\Omega) \times L^2(Q)$. Let $j : \mathbb{R} \to \mathbb{R}^+$ be convex, lower-semi-continuous, and $j(0) = 0$. If $\partial m$ is a function, then $C_2$ is also single-valued and*

$$(3.2) \qquad \left(C_2[u_1, U_1] - C_2[u_2, U_2], [\sigma_1, \sigma_2]\right)_{L^2(\Omega)\times L^2(Q)} \geq 0$$

*for any selections $\sigma_1 \in \partial j(u_1 - u_2)$ in $L^2(\Omega)$ and $\sigma_2 \in \partial j(U_1 - U_2)$ in $L^2(Q)$.*

*Proof.* To show $C_2$ is maximal monotone it suffices to show that for any pair $[f, F] \in L^2(\Omega) \times L^2(Q)$ there is a solution of

$$(3.3.a) \qquad u \in L^2(\Omega) \cap W_0^{1,p}(\Omega) : u + \mathcal{A}(u) - \lambda'(\mu) = f \text{ in } W^{-1,p'}(\Omega)\ ,$$

$$(3.3.b) \qquad U \in L^2(Q) \cap W_q : U + \mathcal{B}(U) + \gamma'(\mu) = F \text{ in } W_q'$$

$$(3.3.c) \qquad \mu \in L^{q'}(\Omega, L^{q'}(\Gamma_x)) : \mu \in \partial\tilde{m}(\gamma U - \lambda u)\ .$$

The existence of a (unique) solution of (3.3) follows as in Proposition 1, but by considering the pseudo-monotone operator $[\mathcal{A}, \mathcal{B}]$ on the product space $L^2(\Omega) \cap W_0^{1,p}(\Omega) \times L^2(Q) \cap W_q$ and the convex function, $\frac{1}{2}\|u\|^2_{L^2(\Omega)} + \frac{1}{2}\|U\|^2_{L^2(Q)} + \tilde{m}(\gamma U - \lambda u)$, on that space.

To establish the estimate (3.2), we consider the lower-semi-continuous convex function

$$(3.4) \quad \tilde{j}[u, U] = \int_\Omega \left( j(u(x)) + \int_{\Omega_x} j(U(x,y))\, dy \right) dx\ , \quad [u, U] \in L^2(\Omega) \times L^2(Q)\ .$$

The subgradient of $\tilde{\jmath}$ is given on this product space by

$$\tilde{\sigma} = [\sigma_1, \sigma_2] \in \partial\tilde{\jmath}[u, U] \text{ if and only if}$$

$$\tilde{\sigma}[v, V] = \int_\Omega \left( \sigma_1(x)v(x) + \int_{\Omega_x} \sigma_2(x, y)V(x, y)\, dy \right) dx\,, \quad [v, V] \in L^2(\Omega) \times L^2(Q))\,,$$

where

$$\sigma_1(x) \in \partial j\big(u(x)\big), \text{ a.e. } x \in \Omega\,,\ \sigma_2(x, y) \in \partial j\big(U(x, y)\big)\,, \text{ a.e. } (x, y) \in Q\,.$$

The Yoshida approximation $\tilde{\jmath}_\varepsilon$ of $\tilde{\jmath}$ is given as in (3.4) but with $j$ replaced by $j_\varepsilon$. Since the derivative of $j_\varepsilon$ is Lipschitz, monotone, and contains the origin, it follows by Lemma 3 that the special case of (3.2) with $j_\varepsilon$ is true. Thus, $C_2$ is $\partial\tilde{\jmath}$-monotone [7] and the desired result follows, since the single-valued $C_2$ equals its minimal section.

We define the realization of (2.6) in $L^r(\Omega) \times L^r(Q)$, $1 \le r < \infty$, as follows. For $r \ge 2$, $C_r$ is the restriction of $C_2$ to $L^r(\Omega) \times L^r(Q)$, and for $1 \le r < 2$, $C_r$ is the closure in $L^r(\Omega) \times L^r(Q)$ of $C_2$.

**Corollary 3.** *The operator $C_r$ is $m$-accretive in $L^r(\Omega) \times L^r(Q)$ for $1 \le r < \infty$.*

*Proof.* Let $(I + \varepsilon C_2)([u_j, U_j]) \ni [f_j, F_j]$, $j = 1, 2$, and assume $[f_j, F_j] \in L^r(\Omega) \times L^r(Q)$ if $r \ge 2$. Set $j(s) = |s|^r$, $s \in \mathbb{R}$. From Proposition 4.7 of [7] it follows that

$$\|[u_1 - u_2, U_1 - U_2]\|_{L^r(\Omega)\times L^r(Q)} \le \|[f_1 - f_2, F_1 - F_2]\|_{L^r(\Omega)\times L^r(Q)}\,.$$

Taking $[f_2, F_2] = [0, 0]$ we see that $L^r(\Omega) \times L^r(Q)$ is invariant under $(I + \varepsilon C_2)^{-1}$, and then the estimate shows this operator is a contraction on that space. We have $Rg(I + \varepsilon C_r) = L^r(\Omega) \times L^r(Q)$ directly from the definition for $r \ge 2$, and for $1 \le r < 2$, $Rg(I + \varepsilon C_r) \supset L^2(\Omega) \times L^2(Q)$, which is dense, so the result follows easily.

*Remarks.* The Cauchy-Dirichlet problem for the regularized model (1.1) is well-posed in $L^r(\Omega) \times L^r(Q)$ when $a(u) = u$, $b(U) = U$, and $r > 1$. This follows from Corollary 3 and the theory of evolution equations generated by $m$-accretive operators in a uniformly convex Banach space. For example, from [18] we recall the following:

*if* $\mathfrak{f} \in W^{1,1}(0,T;X)$ *and* $w_0 \in D(\mathrm{C}_r)$, *where* $\mathrm{C}_r$ *is m-accretive on the uniformly convex Banach space* $X$, *then there exists a unique Lipschitz function* $w : [0,T] \to X$ *for which*

$$w'(t) + \mathrm{C}_r\big(w(t)\big) \ni \mathfrak{f}(t) \;, \qquad \text{a.e. } t \in (0,T) \;,$$

$$w(t) \in D(\mathrm{C}_r) \;\; \textit{for all} \;\; t \in [0,T] \;, \;\; \textit{and}$$

$$w(0) = w_0 \;.$$

See [3] for details (Theorem III.2.3) and references. By applying this result to the operator $\mathrm{C}_r$ given in $X \equiv L^r(\Omega) \times L^r(Q)$, $1 < r < \infty$, we obtain a generalized strong solution $w(t) = [u(t), U(t)]$ of the system

$$\frac{\partial u(x,t)}{\partial t} + Au(x,t) + \int_{\Omega_x} \frac{\partial U(x,y,t)}{\partial t}\,dy = f(x,t) + \int_{\Omega_x} F(x,y,t)\,dy \;, \; x \in \Omega \;, \; t \in (0,T) \;,$$

$$u(s,t) = 0 \;, \qquad s \in \Gamma \;,$$

$$\frac{\partial U(x,y,t)}{\partial t} + BU(x,y,t) = F(x,y,t) \;, \qquad y \in \Omega_x \;,$$

$$\mu(x,s,t) \in \partial m\big(U(x,s,t) - u(x,t)\big) \;, \; \partial_x U(x,s,t) + \mu(x,s,t) = 0 \;, \quad s \in \Gamma_x \;,$$

$$u(x,0) = u_0(x) \;, \quad U(x,y,0) = U_0(x,y) \;.$$

The restrictions on the data $\mathfrak{f}(t) = [f(t), F(t)]$ and $w_0 = [u_0, U_0]$ can be considerably relaxed in the Hilbert space case $r = 2$ [7].

By applying Proposition 1′ similarly, it follows that corresponding results for the matched model are obtained. Thus one obtains a generalized strong solution in $L^r(\Omega) \times L^r(Q)$, $1 < r < \infty$, of the system

$$\frac{\partial u(x,t)}{\partial t} + Au(x,t) + \int_{\Omega_x} \frac{\partial U(x,y,t)}{\partial t}\, dy = f(x,t) + \int_{\Omega_x} F(x,y,t)\, dt \ , \ x \in \Omega \ , \ t \in (0,T) \ ,$$

$$u(s,t) = 0 \qquad s \in \Gamma \ ,$$

$$\frac{\partial U(x,y,t)}{\partial t} + BU(x,y,t) = F(x,y,t) \ , \qquad y \in \Omega_x \ ,$$

$$U(x,s,t) = u(x,t) \ , \qquad s \in \Gamma_x \ ,$$

$$u(x,0) = u_0(x) \ , \quad U(x,y,0) = U_0(x,y) \ .$$

This follows as above from the analog of Proposition 3 and Corollary 3.

We return to consider the fully nonlinear model (1.1). The generator of this evolution system will be obtained by closing up the composition of $\mathrm{C}_2$ with the inverse of $[\partial\varphi, \partial\Phi]$ in $L^1(\Omega) \times L^1(Q)$. Thus, we begin with the following.

*Definition.* $\mathrm{C}[a,b] \ni [f,F]$ if $\mathrm{C}_2[u,U] \ni [f,F]$ and $a \in \partial\varphi(u)$ in $L^2(\Omega)$, $b \in \partial\Phi(U)$ in $L^2(Q)$ for some pair $[u,U]$ as in (3.1).

**Lemma 4.** *The operator* $\mathrm{C}$ *is accretive on* $L^1(\Omega) \times L^1(Q)$ *if either* $\partial m$ *is a function or if both* $\partial\varphi$ *and* $\partial\Phi$ *are functions.*

*Proof.* Let $\varepsilon > 0$ and suppose that $(I + \varepsilon \mathrm{C})[a_j, b_j] \ni [f_j, F_j]$ for $j = 1,2$. Thus we have $\varepsilon \mathrm{C}_2[u_j, U_j] \ni [f_j - a_j, F_j - b_j]$, $a_j \in \partial\varphi(u_j)$, $b_j \in \partial\Phi(U_j)$ as above. First we choose $\sigma(s) = \mathrm{sgn}_\delta^+(s)$, the Yoshida approximation of the maximal monotone $\mathrm{sgn}^+$, apply Lemma 3 and obtain

$$\begin{aligned}\big(a_1 - a_2, \mathrm{sgn}_\delta^+(u_1 - u_2)\big)_{L^2(\Omega)} &+ \big(b_1 - b_2, \mathrm{sgn}_\delta^+(U_1 - U_2)\big)_{L^2(Q)} \\ &\le \|(f_1 - f_2)^+\|_{L^1(\Omega)} + \|(F_1 - F_2)^+\|_{L^1(Q)} \ .\end{aligned}$$

If $\partial\varphi$ and $\partial\Phi$ are functions, then

$$(a_1 - a_2)\,\mathrm{sgn}_0^+(u_1 - u_2) = (a_1 - a_2)^+ \ ,$$

$$(b_1 - b_2)\,\mathrm{sgn}_0^+(U_1 - U_2) = (b_1 - b_2)^+ \ ,$$

so letting $\delta \to 0$ gives

$$\|(a_1 - a_2)^+\|_{L^1(\Omega)} + \|(b_1 - b_2)^+\|_{L^1(Q)} \le \|(f_1 - f_2)^+\|_{L^1(\Omega)} + \|(F_1 - F_2)^+\|_{L^1(Q)} \ . \tag{3.5}$$

The same holds for negative parts, so it follows that $(I+\varepsilon\mathrm{C})^{-1}$ is an order-preserving contraction with respect to $L^1(\Omega)\times L^1(Q)$ for each $\varepsilon>0$.

Next we suppose $\partial m$ is a function. Choose $j(s)=s^+$, so that $\partial j=\mathrm{sgn}^+$, and then set

$$\sigma_1(x)=\mathrm{sgn}_0^+(u_1-u_2+a_1-a_2)\in\mathrm{sgn}^+(u_1-u_2)\cap\mathrm{sgn}^+(a_1-a_2)\,,$$
$$\sigma_2(x,y)=\mathrm{sgn}_0^+(U_1-U_2+b_1-b_2)\in\mathrm{sgn}^+(U_1-U_2)\cap\mathrm{sgn}^+(b_1-b_2)\,.$$

Proposition 3 applies here to give (3.5). A similar estimate for negative parts yields the result.

Although C is not accretive on $L^r$ for $1<r$, we can obtain $L^\infty$ estimates when the graphs $\partial\varphi$, $\partial\Phi$ are not too dissimilar.

**Corollary 4.** *If $(I+\varepsilon\mathrm{C})[a,b]\ni[f,F]$ with $\varepsilon>0$, then*

$$\|a^+\|_{L^\infty(\Omega)}\le\max\big(a_0(k),\|f^+\|_{L^\infty(\Omega)}\big)\,,$$
$$\|b^+\|_{L^\infty(Q)}\le\max\big(b_0(k),\|F^+\|_{L^\infty(Q)}\big)\,, \tag{3.6}$$

*where $k\equiv\max(a_0^{-1}(\|f^+\|_{L^\infty}),b_0^{-1}(\|F^+\|))$.*

*Remarks.* Here $a_0$ is the *minimal section* $(\partial\varphi)_0$, $a_0^{-1}$ is the minimal section of $(\partial\varphi)^{-1}$, and $b_0,b_0^{-1}$ are defined similarly from $\partial\Phi$. Specifically, we obtain an explicit a-priori bound on $\|a^+\|_{L^\infty(\Omega)}$ and $\|b^+\|_{L^\infty(Q)}$ when $\|f^+\|_{L^\infty(\Omega)}\in Rg(\partial\varphi)$ and $\|F^+\|_{L^\infty(Q)}\in Rg(\partial\Phi)$. By similar estimates for negative parts, we obtain explicit estimates on $\|a\|_{L^\infty(\Omega)}$ and $\|b\|_{L^\infty(Q)}$ for *any* pair $f\in L^\infty(\Omega)$, $F\in L^\infty(\Omega)$ if $Rg(\partial\varphi)=\mathbb{R}$ and $Rg(\partial\Phi)=\mathbb{R}$ or (trivially) if both $Rg(\partial\varphi)$ and $Rg(\partial\Phi)$ are bounded in $\mathbb{R}$. Finally, we note that in the special case $\varphi=\Phi$, we obtain

$$\max\big(\|a^+\|_{L^\infty(\Omega)},\|b^+\|_{L^\infty(Q)}\big)\le\max\big(\|f^+\|_{L^\infty(\Omega)},\|F^+\|_{L^\infty(Q)}\big)\,.$$

*Proof.* By the choice of $k\ge0$ we have

$$\partial\varphi(k)\ni\ell_1\ge\|f^+\|_{L^\infty}\quad,\quad\partial\Phi(k)\ni\ell_2\ge\|F^+\|_{L^\infty}\qquad.$$

for some pair $\ell_1, \ell_2$. Subtract these from the operator equation, multiply by either

$$\mathrm{sgn}_\delta^+(u-k) \quad , \quad \mathrm{sgn}_\delta^+(U-k)$$

or by

$$\mathrm{sgn}_0^+(a-\ell_1+u-k) \quad , \quad \mathrm{sgn}_0^+(b-\ell_2+U-k) ,$$

depending on whether $\partial\varphi$ and $\partial\Phi$ are functions or $\partial m$ is a function, respectively. Apply Lemma 3 and let $\delta \to 0$ or apply Proposition 3, respectively, to obtain

$$\|(a-\ell_1)^+\|_{L^1(\Omega)} + \|(b-\ell_2)^+\|_{L^1(Q)} \le \|(f^+-\ell_1)^+\|_{L^1(\Omega)} + \|(F^+-\ell_2)^+\|_{L^1(Q)} .$$

The right side is zero, so the result follows.

**Proposition 4.** (Moser) *Let $(u,U) \in W_0^{1,p}(\Omega) \times \mathcal{W}_q$ be a solution to*

$$\mathcal{A}(u) - \lambda'\mu \ni f \quad in \quad W^{-1,p'}(\Omega) ,$$
$$\mathcal{B}(U) + \gamma'\mu \ni F \quad in \quad \mathcal{W}_q' ,$$
$$\mu \in \partial m(\gamma U - \lambda u) .$$

*1) If $(f,F) \in L^{r'}(\Omega) \times L^{r'}(Q)$ with $r' > \frac{n}{p}$, and*

(2.2.c′) $$\widetilde{A}(x,\vec{\xi}) \cdot \vec{\xi} \ge c_0|\xi|^p - g_0(x)$$

*where $g_0 \in L^{r'}(\Omega)$, then $u \in L^\infty(\Omega)$.*

*2) If, additionally, $F \in L^\infty[\Omega; L^{t'}(\Omega_x)]$ with $t' > \frac{n}{q}$,*

(2.3.c′) $$\widetilde{B}(x,y,\vec{\xi}) \cdot \vec{\xi} \ge c_0|\xi|^q - h_0(x,y)$$

*where $h_0 \in L^\infty[\Omega; L^{t'}(\Omega_x)]$, and $m$ satisfies the growth condition (2.5) and $m(0) = 0$, then $U \in L^\infty(Q)$.*

*Proof.*

1) Estimate (2.7) of Proposition 1 shows that $\lambda'\mu \in L^{r'}(\Omega)$, so that

$$\mathcal{A}(u) = f - \lambda'\mu = \tilde{f} \in L^{r'}(\Omega) .$$

Lemma 3 of [22] can now be used to conclude $u \in L^\infty(\Omega)$.

2) Define $\widetilde{U} = U - u1$. Since $\mathcal{B}(\widetilde{U}) = \mathcal{B}(U)$, it follows that

$$\mathcal{B}(\widetilde{U}) + \gamma'\mu = F \text{ in } \mathcal{W}_q', \qquad \mu \in \partial m(\gamma\widetilde{U}),$$

and for almost every $x \in \Omega$, and every $V \in \mathcal{W}_q$

$$(*) \qquad \int_{\Omega_x} \widetilde{B}(x,\cdot,\nabla_y\widetilde{U}(x))\cdot\nabla_y V(x) + \int_{\Gamma_x} \mu(x)\gamma V(x) = \int_{\Omega_x} F(x,\cdot)V(x),$$

with $\mu(x) \in \partial m(\gamma\widetilde{U}(x))$. We now use Moser iteration with $(*)$ to conclude $\|\widetilde{U}(x)\|_{L^\infty(\Omega_x)} \leq C$, where $C$ is to be chosen independently of $x \in \Omega$.

If $\widetilde{U}(x) \in L^r(\Omega_x)$ ($r = q$ suffices for the first iterate), define $s = 1 + \frac{r-t}{tq}$ ($\frac{1}{t}+\frac{1}{t'} = 1$). Let $H \in C^1(\mathbb{R})$ satisfy $H(s) = |s|^s$ if $|s| \leq s_0$, $H$ affine for $|s| > s_0$, and define $G(s) = \int_0^s |H'(\xi)|^q \, d\xi$. Since $H$ has linear growth, it follows that $G(\widetilde{U}) \in \mathcal{W}_q$. Substituting $G(\widetilde{U})$ for $V$ in $(*)$ gives

$$\int_{\Omega_x} \widetilde{B}(x,\cdot,\nabla_y\widetilde{U}(x))\cdot\nabla_y\widetilde{U}(x)G'(\widetilde{U}(x)) + \int_{\Gamma_x} \mu(x)\gamma G(\widetilde{U}(x)) = \int_{\Omega_x} F(x,\cdot)G(\widetilde{U}(x)).$$

The first term of the above is bounded below using (2.3.c$'$). To estimate the second term, use

(i) $\mu\widetilde{U} \geq m(\widetilde{U})$ (as $m(0) = 0$), and

(ii) $\operatorname{sgn}(\widetilde{U}) = \operatorname{sgn}(G(\widetilde{U}))$ (so that $G(\widetilde{U})/\widetilde{U} \geq 0$ when $\widetilde{U} \neq 0$),

to get

$$\begin{aligned}\mu G(\widetilde{U}) &= \mu\widetilde{U}\ G(\widetilde{U})/\widetilde{U} \geq m(\widetilde{U})\ G(\widetilde{U})/\widetilde{U} \\ &\geq c_0|\widetilde{U}|^q\ G(\widetilde{U})/\widetilde{U} = c_0|\widetilde{U}|^{q-1}|G(\widetilde{U})|.\end{aligned}$$

$$c_0\int_{\Omega_x} |\nabla_y\widetilde{U}|^q G'(\widetilde{U}) + c_0\int_{\Gamma_x} |\widetilde{U}|^{q-1}|G(\widetilde{U})| \leq \int_{\Omega_x} FG(\widetilde{U}) + h_0 G'(\widetilde{U}).$$

The first term may be written as $|\nabla_y H(\widetilde{U})|^q$ which, using the Sobolev embedding theorem, is bounded below by

$$c(\varepsilon)\|H(\widetilde{U})\|^q_{L^{\frac{nq}{n-q}}(\Omega_x)} - \varepsilon\int_{\Gamma_x} |H(\widetilde{U})|^q,$$

where $\varepsilon > 0$ can be chosen arbitrarily small (see equation (2.9)). The right hand side is bounded using Hölder's inequality.

$$c(\varepsilon)\|H(\widetilde{U})\|^q_{L^{\frac{nq}{n-q}}(\Omega_x)} + \int_{\Gamma_x} |\widetilde{U}|^{q-1}|G(\widetilde{U})| - \varepsilon|H(\widetilde{U})|^q$$
$$\leq \frac{1}{c_0}\left(\|F\|_{L^{t'}(\Omega_x)}\|G(\widetilde{U})\|_{L^t(\Omega_x)} + \|h_0\|_{L^{t'}(\Omega_x)}\|G'(\widetilde{U})\|_{L^t(\Omega_x)}\right)$$

when $s_0 \to \infty$, $H(\widetilde{U}) \to |\widetilde{U}|^s$, and $|\widetilde{U}|^{q-1}|G(\widetilde{U})| \to \eta(r)|\widetilde{U}|^{sq}$, where $\eta(r) = \frac{t}{r}s^q = \frac{t}{r}(1+\frac{r-t}{qt})^q$. If $\varepsilon$ is chosen as $\varepsilon = \min_{q\leq r<\infty}\eta(r)$, it follows that

$$\|\widetilde{U}\|_{L^{sq(n/n-q)}(\Omega_x)} \leq (cs)^{1/s}\max[1, \|\widetilde{U}\|_{L^r(\Omega_x)}] .$$

The result now follows by iteration of the above estimate.

**Theorem 1.** *Assume the hypotheses of Proposition 1, Corollary 1, Lemma 4, and Propostion 4. Also, assume that $Rg(\partial\varphi)$ and $Rg(\partial\Phi)$ are both bounded or that both are equal to $\mathbb{R}$. Then $\overline{\mathrm{C}}$, the closure of $\mathrm{C}$ in $L^1(\Omega) \times L^1(Q)$, is m-accretive.*

*Proof.* Let $f \in L^\infty(\Omega)$ and $F \in L^\infty(Q)$. Corollary 1 asserts there is a solution of (2.12). If the graphs $\partial\varphi$ and $\partial\Phi$ have bounded range, then $a \in L^\infty(\Omega)$, $b \in L^\infty(Q)$, and it follows from Proposition 4 that $u \in L^2(\Omega)$ and $U \in L^2(Q)$. This shows $\mathrm{C}_2[u, U] \ni [a - f, b - F]$, so $(I + \mathrm{C})([a, b]) \ni [f, F]$. Thus, $Rg(I + \overline{\mathrm{C}})$ is dense in and, hence, equal to $L^1(\Omega) \times L^1(Q)$.

If the ranges of $\partial\varphi$ and $\partial\Phi$ equal $\mathbb{R}$, then by Corollary 4 any solution satisfies

$$\|a\|_{L^\infty(\Omega)} \leq K \quad , \quad \|b\|_{L^\infty(Q)} \leq K ,$$

where $K$ depends on $f$ and $F$. Replace $\partial\varphi, \partial\Phi$ by the appropriately truncated $\partial\varphi_K, \partial\Phi_K$ The solution with these truncated graphs then is a solution of the equation with the original graphs, so we are done.

**Corollary 5.** *Under the hypotheses of Theorem 1, problem (1.1) has a unique generalized solution $(a, b) \in C[0, T; L^1(\Omega) \times L^1(Q)]$, provided the data satisfy $(f, F) \in L^1[0, T; L^1(\Omega) \times L^1(Q)]$, and $(a(0), b(0)) \in \overline{D(\mathrm{C})}$.*

This follows from the Crandall-Liggett Theorem [8], which is proved by showing that the step functions, $(a^N, b^N)$, constructed from solutions to the differencing scheme

$$(3.7) \qquad (a^n, b^n) - (a^{n-1}, b^{n-1}) + \tau \mathbb{C}(a^n, b^n) \ni \tau(f^n, F^n)$$

($\tau = \frac{T}{N}$), converge uniformly when the operator $\mathbb{C}$ is $m$-accretive. Benilan [5] proves that these generalized solutions are unique.

All of our results hold for the matched microstructure model problem. Specifically, Lemma 4 and Corollary 4 are obtained from Proposition 3, and Proposition 4 is actually simpler for the matched problem. The analogs of Theorem 1 and Corollary 5 show that the matched problem (1.1.a), (1.1.b), (1.1.c$'$) has a unique generalized solution $(a, b) \in C[0, T; L^1(\Omega) \times L^1(Q)]$.

The next theorem shows that if the data is further restricted, the generalized solutions will satisfy the partial differential equation (1.1). The following notation is used,

$$\begin{aligned}
&L^r(T) = L^r[0, T; L^r(\Omega) \times L^r(Q)] \qquad 1 \le r \le \infty \,, \\
&V = W_0^{1,p}(\Omega) \times \mathcal{W}_q \,, \\
&\mathcal{V}(T) = L^p[0, T; W_0^{1,p}(\Omega)] \times L^q[0, T; \mathcal{W}_q] \,, \\
&\widehat{\mathcal{V}}(T) = W^{1,p'}[0, T; W^{-1,p'}(\Omega)] \times W^{1,q'}[0, T; \mathcal{W}_q'] \,.
\end{aligned}$$

**Theorem 2.** *Assume the hypotheses of Theorem 1 and in addition that $(f, F) \in L^1(T) \cap \mathcal{V}(T)'$ and $(a(0), b(0)) \in \overline{D(\mathbb{C})} \cap V'$. Then the generalized solutions of Corollary 5 satisfy*

$$(3.8.a) \qquad (a, b) \in \widehat{\mathcal{V}}(T) \,, \quad (u, U) \in \mathcal{V}(T) \,,$$

$$(3.8.b) \qquad \frac{\partial}{\partial t}(a, b) + \left(\mathcal{A}(u) - \lambda' \mu, \mathcal{B}(U) + \gamma' \mu\right) = (f, F) \text{ in } \mathcal{V}(T)' \,,$$

$$(3.8.c) \qquad (a, b) \in \left(\partial\phi(u), \partial\Phi(U)\right) \,, \quad \mu \in \partial\tilde{m}(\lambda u - \gamma U) \,.$$

*Proof.* The results of Grange and Mingot [14] show that the step functions $(a^N, b^N)$ and $(u^N, U^N)$ generated from the differencing scheme (3.7) converge weakly in $\widehat{\mathcal{V}}(T)$

and $\mathcal{V}(T)$ respectively. Moreover, equation (3.8) will be satisfied in the limit, provided the weak limits $(a, b)$ and $(u, U)$ satisfy $(a, b) \in (\partial\phi(u), \partial\Phi(U))$. To establish this inclusion, let $(v, V) \in \mathcal{V}(T)$ and $(\tilde{a}, \tilde{b}) \in (\partial\phi(v), \partial\Phi(V))$. The growth conditions on $\phi$ and $\Phi$ guarantee that $(a^N, b^N)$ and $(\tilde{a}, \tilde{b}) \in \mathcal{V}(T)'$ are functions, so it is possible to define $(a^N - \tilde{a},\, b^N - \tilde{b})_s$ to be the pair of functions truncated above and below by $\pm s$ $(s > 0)$. This pair of functions is bounded in $L^\infty(T)$ and converges in $L^1(T)$ to $(a - \tilde{a},\, b - \tilde{b})_s$, so converges in $L^r(T)$ for $1 \le r < \infty$. If $r \ge \max(p', q')$, it follows that $L^r(T) \subset \mathcal{V}(T)'$, so the sequence $(a^N - \tilde{a}, b^N - \tilde{b})_s$ converges strongly in $\mathcal{V}(T)'$. The monotonicity of $\partial\phi$ and $\partial\Phi$ imply

$$0 \le \left\langle (a^N - \tilde{a}, b^N - \tilde{b})_s \, , \, (u^N - v,\, U^n - V) \right\rangle .$$

Passing to the limit as $N \to \infty$, and then letting $s \to \infty$ yields

$$0 \le \left\langle (a - \tilde{a},\, b - \tilde{b}) \, , \, (u - v,\, U - V) \right\rangle , \qquad (\tilde{a}, \tilde{b}) \in \big(\partial\phi(v), \partial\Phi(V)\big) .$$

Since $(\partial\phi(\cdot), \partial\Phi(\cdot))$ is maximally monotone, it follows that $(a, b) \in (\partial\phi(u), \partial\Phi(U))$. ∎

Finally, we note that the corresponding solution of the matched problem satisfies

(3.8.a′) $\qquad (a, b) \in \widetilde{\mathcal{V}}(T) \, , \quad (\gamma U, U) \in \mathcal{V}(T) \, ,$

(3.8.b′) $\qquad \dfrac{\partial}{\partial t}(a, b) + \big(\mathcal{A}(\gamma U), \mathcal{B}(U)\big) = (f, F) \;\text{ in }\; \mathcal{V}_0(T)' \, ,$

(3.8.c′) $\qquad (a, b) \in \big(\partial\varphi(\gamma U), \partial\Phi(U)\big) \, , \quad U \in \mathcal{W}_0 \, ,$

where the space $\mathcal{V}_0(T)$ is given by

$$\mathcal{V}_0(T) \equiv \left\{ U \in L^q[0, T; \mathcal{W}_1] : \gamma(U) \in L^p\left[0, T; W_0^{1,p}(\Omega)\right] \right\}$$

with the appropriate norm for which $(\gamma(U), U) \in \mathcal{V}(T)$ for each $U \in \mathcal{V}_0(T)$.

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# Diffusion in a Fissured Medium with Micro-Structure


R.E. SHOWALTER*

Department of Mathematics
The University of Texas at Austin
Austin, Texas 78712


The work described below was done with N.J. Walkington of Carnegie-Mellon University. We establish that the Cauchy-Dirichlet problem for degenerate parabolic systems of the form

$$\text{(1.a)}\quad \frac{\partial}{\partial t}a(u) - \vec{\nabla}\cdot\widetilde{A}(x,\vec{\nabla}u) + \int_{\Gamma_x}\widetilde{B}(x,s,\vec{\nabla}_y U)\cdot\vec{\nu}\,ds \ni f\ , \qquad x\in\Omega\ ,$$

$$\text{(1.b)}\quad \frac{\partial}{\partial t}b(U) - \vec{\nabla}_y\cdot\widetilde{B}(x,y,\vec{\nabla}_y U) \ni F\ , \qquad y\in\Omega_x\ ,$$

$$\text{(1.c)}\quad \widetilde{B}(x,y,\vec{\nabla}_y U)\cdot\vec{\nu} + \mu\big(U(x,y,t) - u(x,t)\big) \ni 0\ , \qquad y\in\Gamma_x\ ,$$

is well-posed. Here $\Omega$ is a domain in $\mathbf{R}^n$ and for each value of the macro-variable $x \in \Omega$ is specified a domain $\Omega_x$ with boundary $\Gamma_x$ for the micro-variable $y \in \Omega_x$. Each of $a, b, \mu$ is a maximal monotone graph. These graphs are not necessarily strictly increasing; they may be piecewise constant or multi-valued. The elliptic operators in (1.a) and (1.b) are non-linear in the gradient of degree $p-1>0$ and $q-1>0$, respectively, with $\frac{1}{q}+\frac{1}{n}\geq\frac{1}{p}$, so some specific degeneracy is also permitted here. Certain first order spatial derivatives can be added to (1.a) and (1.b) with no difficulty, and corresponding problems with constraints, *i.e.*, variational inequalities, can be treated similarly. A particular example important for


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applications is the linear constraint

$$(1.c') \qquad U(x,y,t) = u(x,t)\ , \qquad y \in \Gamma_x\ ,\ x \in \Omega$$

which then replaces (1.c). The system (1) with $\mu(s) = \frac{1}{\varepsilon}|s|^{q-2}s$ is called a *regularized micro-structure model*, and (1.a), (1.b), (1.c′) is the corresponding *matched micro-structure model* in which (formally) $\varepsilon \to 0$. An example of such a system is a model for the flow of a fluid (liquid or gas) through a fractured medium. In such a context, (1.a) prescribes the flow on the global scale of the fissure system and (1.b) gives the flow on the microscale of the individual cell at a specific point $x$ in the fissure system. The transfer of fluid between the cells and surrounding medium is prescribed by (1.c) or (1.c′). A major objective is to accurately model this fluid exchange between the cells and fissures.

Systems of the form (1) were developed in [10], [11], [3] in physical chemistry as models for diffusion through a medium with a prescribed microstructure. Similar systems arose in soil science [2], [6] and in reservoir models for fractured media [4], [7]. By homogenization methods such systems are obtained as limits of exact micro-scale models, and then the effective coefficients are computed explicitly from local material properties [13], [8], [1]. An existence-uniqueness theory for these linear problems which exploits the strong parabolic structure of the system was given in [12]. One can alternatively eliminate $U$ and obtain a single functional differential equation for $u$ in the simpler space $L^2(\Omega)$, but the structure of the equation then obstructs the optimal parabolic type results [9]. Also see [5] for a nonlinear system with reaction-diffusion local effects.

We begin by stating and resolving the stationary forms of our systems. Denote by $\gamma_x : W^{1,q}(\Omega_x) \to L^q(\Gamma_x)$ the *trace* map which assigns boundary values. Let $T_x$ be the range of $\gamma_x$; this is a Banach space with the norm induced by $\gamma_x$ from $W^{1,q}(\Omega_x)$. Since $\Gamma_x$ is smooth, there is a unit outward normal $\nu_x(s)$ at each $s \in \Gamma_x$. We shall denote by $\vec{\nabla}_y$ the gradient on $W^{1,q}(\Omega_x)$ and by $\vec{\nabla}$ the gradient on $W^{1,p}(\Omega)$.

In order to prescribe a measurable family of cells, $\{\Omega_x, x \in \Omega\}$, let $Q \subset \Omega \times \mathbf{R}^n$ be a given measurable set, and set $\Omega_x = \{y \in \mathbf{R}^n : (x,y) \in Q\}$. Each $\Omega_x$ is measurable in $\mathbf{R}^n$ and by zero-extension we identify $L^q(Q) \hookrightarrow L^q(\Omega \times \mathbf{R}^n)$ and each $L^q(\Omega_x) \hookrightarrow L^q(\mathbf{R}^n)$. Thus we obtain from above

$$L^q(Q) \cong \left\{U \in L^q(\Omega, L^q(\mathbf{R}^n)) : U(x) \in L^q(\Omega_x) \text{ , a.e. } x \in \Omega\right\} .$$

We shall denote the duality on this Banach space by

$$(U, \Phi)_{L(Q)} = \int_\Omega \left\{ \int_{\Omega_x} U(x,y)\Phi(x,y)\,dy \right\} dx \ , \quad U \in L^q(Q) \ , \ \Phi \in L^{q'}(Q) \ .$$

The *state space* for our problems will be the product $L^1(\Omega) \times L^1(Q)$.

Note that $W^{1,q}(\Omega_x)$ is continuously imbedded in $L^q(\Omega_x)$, uniformly for $x \in \Omega$. It follows that the direct sum

$$\mathcal{W}_q \equiv L^q(\Omega, W^{1,q}(\Omega_x)) \equiv \ \{U \in L^q(Q) : U(x) \in W^{1,q}(\Omega_x) \text{ , a.e. } x \in \Omega \ ,$$

$$\text{and } \int_\Omega \|U(x)\|^q_{W^{1,q}}\, dx < \infty\}$$

is a Banach space. We use a variety of such spaces which can be constructed in this manner. The Banach space $W_0^{1,p}(\Omega) \times \mathcal{W}_q$ will be the *energy space* for the regularized problem (1). We assume the trace maps $\gamma_x : W^{1,q}(\Omega_x) \to L^q(\Gamma_x)$ are *uniformly* bounded. Thus for each $U \in \mathcal{W}_q$ it follows that the *distributed trace* $\gamma(U)$ defined by $\gamma(U)(x,s) \equiv (\gamma_x(U(x))(s)$, $s \in \Gamma_x$, $x \in \Omega$, belongs to $L^q(\Omega, L^q(\Gamma_x))$. The distributed trace $\gamma$ maps $\mathcal{W}_q$ onto $\mathcal{T}_q \equiv L^q(\Omega, T_x) \hookrightarrow L^q(\Omega, L^q(\Gamma_x))$.

Denote by $\lambda$ the map of $L^q(\Omega)$ into $\mathcal{T}_q$ given by $\lambda v(x) = v(x) \cdot \mathbf{1}_x$, a.e. $x \in \Omega$; $v \in L^q(\Omega)$; $\lambda$ is an isomorphism of $L^q(\Omega)$ onto a closed subspace of $\mathcal{T}_q$. Tne dual map $\lambda'$ taking $\mathcal{T}_q'$ into $L^{q'}(\Omega)$ is given by

$$\lambda' g(v) = g(\lambda v) = \int_\Omega g_x(\mathbf{1}_x) \cdot v(x)\,dx \ , \qquad g \in \mathcal{T}_q' \ , \ v \in L^q(\Omega) \ ,$$

so we have $\lambda' g(x) = g_x(1_x)$, a.e. $x \in \Omega$.

We consider elliptic differential operators in divergence form as realizations of monotone operators from Banach spaces to their duals. The global diffusion operator $\mathcal{A} : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is given by

$$\mathcal{A}u(v) = \int_\Omega \widetilde{A}(x, \vec{\nabla}u(x))\vec{\nabla}v(x)\,dx\ , \qquad u, v \in W_0^{1,p}(\Omega)\ .$$

Similarly define for each $x \in \Omega$

$$\mathcal{B}_x w(v) = \int_{\Omega_x} \widetilde{B}(x, y, \vec{\nabla}_y w(y))\vec{\nabla}_y v(y)\,dy\ , \qquad w, v \in W^{1,q}(\Omega_x)\ .$$

Also, we shall denote by $\mathcal{B} : \mathcal{W}_q \to \mathcal{W}_q'$ the *distributed* operator constructed from the collection $\{\mathcal{B}_x : x \in \Omega\}$ by

$$\mathcal{B}U(x) = \mathcal{B}_x(U(x))\ , \qquad \text{a.e. } x \in \Omega\ ,\ U \in \mathcal{W}_q\ ,$$

and we note that this is equivalent to

$$\mathcal{B}U(V) \equiv \int_\Omega \mathcal{B}_x(U(x))V(x)\,dx\ , \qquad U, V \in \mathcal{W}_q\ .$$

The coupling term in our system will be given as a monotone graph which is a subgradient operator. Thus, assume $m : \mathbb{R} \to \mathbb{R}^+$ is convex, continuous and bounded by $C(|s|^q + 1)$. Then by

$$\tilde{m}(g) \equiv \int_\Omega \int_{\Gamma_x} m(g(x, s))\,ds\,dx\ , \qquad g \in L^q(\Omega, L^q(\Gamma_x))$$

we obtain the convex, continuous $\tilde{m} : L^q(\Omega, L^q(\Gamma_x)) \to \mathbb{R}^+$.

For the case of the *degenerate stationary problem* corresponding to (1), we obtain the following result.

Let $\varphi : \mathbb{R} \to \mathbb{R}^+$ and $\Phi : \mathbb{R} \to \mathbb{R}^+$ be convex and continuous, with $\varphi(0) = \Phi(0) = 0$, and assume

$$\varphi(s) \le C(|s|^q + 1)\ , \quad \Phi(s) \le C(|s|^q + 1)\ , \qquad s \in \mathbb{R}\ .$$

For each pair $f \in W^{-1,p'}(\Omega)$, $F \in \mathcal{W}_q'$, there exists a solution of

$$\text{(2.a)} \qquad u \in W_0^{1,p}(\Omega) : a + A(u) - \lambda'(\mu) = f \ \text{ in } \ W^{-1,p'}(\Omega)$$

$$\text{(2.b)} \qquad U \in \mathcal{W}_q : b + \mathcal{B}(U) + \gamma'(\mu) = F \ \text{ in } \ \mathcal{W}_q'$$

$$\text{(2.c)} \qquad \mu \in \partial\tilde{m}(\gamma U - \lambda u) \ \text{ in } \ L^{q'}(\Omega, L^{q'}(\Gamma_x)) \ ; \ \text{ and}$$

$$\text{(2.d)} \qquad a \in \partial\varphi(u) \ \text{ in } \ L^{q'}(\Omega) \ , \ b \in \partial\Phi(U) \ \text{ in } \ L^{q'}(Q) \ .$$

For any such solution we have

$$\int_{\Omega_x} b(x,y)\,dy + \int_{\Gamma_x} \mu(x,s)\,ds = \langle F(x), \mathbf{1}_x \rangle \ , \quad \text{a.e. } \ x \in \Omega \ .$$

Next we restrict our system to appropriate products of $L^r$ spaces. The Hilbert space case, $r = 2$, serves not only as a convenient starting point but leads to the generalized accretive estimates we shall need for the singular case of (1) in which $a$ or $b$ is not only nonlinear but multi-valued. The stationary operator for (1) is shown to be $m$-accretive in the $L^1$ space, so we obtain a ***generalized solution*** in the sense of the nonlinear semigroup theory for general Banach spaces. As an intermediate step we show the special case of $a = b =$ identity is resolved as a ***strong solution*** in every $L^r$ space, $1 < r < \infty$, and also in appropriate dual Sobolev spaces.

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008

# NODAL SUPERCONVERGENCE AND SOLUTION ENHANCEMENT FOR A CLASS OF FINITE-ELEMENT AND FINITE-DIFFERENCE METHODS*

R. J. MACKINNON† AND G. F. CAREY†

**Abstract.** A class of finite-element methods for elliptic problems is shown to exhibit nodal superconvergence in the approximate solution, and some equivalence properties to familiar finite-difference operators are demonstrated. The superconvergence property is exploited in a Taylor series analysis to demonstrate Gauss-point superconvergence for the derivatives of the approximation. A post-processing formula for the derivative at the nodes is constructed and shown to exhibit superconvergence. The nodal superconvergence property can be exploited recursively to further enhance the finite-element or finite-difference solution. Supporting numerical studies are given.

**Key words.** finite element, finite difference, superconvergence, post-processing

**AMS(MOS) subject classifications.** 65-L60, 65-N30

**1. Introduction.** In this note we consider a Galerkin finite-element approximation of the Dirichlet problem for the equation $Lu=f$ in $\Omega$. Here $\Omega$ is a union of rectangular subdomains, $L$ is a second-order elliptic differential operator with smooth coefficients, and $u$ is assumed to be sufficiently smooth. By introducing an appropriate integration rule for element quadrature we show that the Galerkin approximation $u_h$, defined on a square mesh of piecewise bilinear elements, is equivalent to a familiar finite-difference approximation of $u$. Discrete uniform error estimates for this difference approximation are known (Bramble and Hubbard [1]). These estimates imply that difference quotients of the error have the same order of convergence as the error itself; i.e., $O(h^2)$ for the bilinear element. It follows that this nodal superconvergence property holds for the standard Galerkin approximation with higher-order (or full) integration. We use this result to prove new superconvergence results and show how simple and accurate superconvergent post-processing formulas for the solution and derivatives can be derived using Taylor series expansions. Although the formulation and analysis presented here is for problems in two dimensions, the results apply to problems in one dimension as well, and extend directly to three dimensions.

**2. Formulation and analysis.**

**2.1. Nodal solution superconvergence.** Consider the boundary value problem

$$Lu=-\left[\frac{\partial}{\partial x}\left(a_1\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(a_2\frac{\partial u}{\partial y}\right)\right]+b_1u_x+b_2u_y+cu=f \tag{1}$$

in the unit square $\Omega=(0,1)\times(0,1)$ with Dirichlet data

$$u=g \quad \text{on } \partial\Omega. \tag{2}$$

Here we assume that $\mathbf{a}$, $\mathbf{b}$, $c$, and $f$ are smooth, and $L$ is uniformly elliptic in $\Omega$.

The Galerkin finite-element approximation to (1) is defined to be $u_h \in H^h$, satisfying the essential boundary condition, and such that

$$B(u_h, v_h)=(f, v_h) \tag{3}$$

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† Department of Aerospace Engineering and Engineering Mechanics, University of Texas, Austin, Texas 78712. This research has been supported in part by the Office of Naval Research.

for all $v_h \in H_0^h \subset H_0^1(\Omega)$, where $(\cdot,\cdot)$ is the $L^2(\Omega)$ inner product and $B(\cdot,\cdot)$ is the bilinear functional

$$B(w,\psi)=\int_\Omega [a_1 w_x\psi_x + a_2 w_y\psi_y + (b_1 w_x + b_2 w_y + cw)\psi]\, dx\, dy. \tag{4}$$

Now consider a uniform partition of $\Omega$ into square elements of size $h$ and take $H^h(\Omega)$ to be spanned by $C^0$ piecewise-bilinear functions defined on this partition. Approximating the integrals in (3) by a suitable integration rule applied over each element, we get the approximation $B_h(u_h, v_h)=(f, v_h)_h$ for all $v_h \in H^h$. The resulting algebraic system is

$$\mathbf{B}_h \mathbf{u}_h = \mathbf{f}_h \tag{5}$$

where the precise forms of $\mathbf{B}_h$ and $\mathbf{f}_h$ depend on the particular integration rule used.

For clarity of exposition, let us first consider the case where coefficients $\mathbf{a}$, $\mathbf{b}$, and $c$ are constants, and a $(2\times 2)$ trapezoidal integration rule is used to evaluate integrals in (3). Accordingly, evaluating the coefficients in (5), for typical interior node point $i$ at $(x_i, y_i)$ with test function $v_{hi}$, we obtain

$$\begin{aligned}B_h(u_h, v_{hi}) = &-\{a_1[u_h(x_i+h, y_i)-2u_h(x_i,y_i)+u_h(x_i-h,y_i)]\\ &+a_2[u_h(x_i, y_i+h)-2u_h(x_i,y_i)+u_h(x_i,y_i-h)]\}\\ &+h\frac{b_1}{2}[u_h(x_i+h,y_i)-u_h(x_i-h,y_i)]\\ &+h\frac{b_2}{2}[u_h(x_i,y_i+h)-u_h(x_i,y_i-h)]+cu_h(x_i,y_i)h^2\end{aligned} \tag{6}$$

and

$$(f, v_{hi})_h = f(x_i, y_i)h^2. \tag{7}$$

For this case, we see from (6) and (7) that (5) is equivalent to the five-point central difference approximation to (1).

Bramble and Hubbard [1] have shown that, for a solution $u$ of (1) having bounded fifth derivatives, the gridpoint error $e_i = u(x_i, y_i) - u_h(x_i, y_i)$ for the five-point difference approximation satisfies

$$e_i = \phi(x_i, y_i)h^2 + R(x_i, y_i, h)h^3 \tag{8}$$

where $\phi$ has Lipschitz continuous second derivatives, and $R$ is uniformly bounded in $x_i$, $y_i$, and $h$. It follows from (8) that the solution to this finite-element problem (5) has gridpoint errors of order $O(h^2)$. It should be emphasized that this estimate is a gridpoint result for the discrete problem, and is of the same order as the global $L^2$ estimate usually encountered in finite-element theory.

*Remark.* If the domain discretization error is zero (as assumed here), then it follows directly from the proof in Bramble and Hubbard that $\max_{x_i, y_i} |R(x_i, y_i, h)| \leqq Ch$, constant $C$, so the final term in (8) is actually $O(h^4)$.

In (8) $\phi$ is the solution to the auxiliary problem

$$\begin{aligned} L\phi &= \tau(u) \quad \text{in } \Omega,\\ \phi &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{9}$$

with $\tau$ the truncation error. In particular,

$$B_h(u, v_{hi}) = h^2 Lu(x_i, y_i) + h^4\tau_i(u) + O(h^6) \tag{10}$$

where $\tau_i(u)$ denotes $\tau(u)$ evaluated at interior gridpoint $(x_i, y_i)$. For trapezoidal integration $\tau(u)$ corresponding to (6) is

(11) $$\tau(u) = -\tfrac{1}{12}[a_1 u_{xxxx} + a_2 u_{yyyy} - 2(b_1 u_{xxx} + b_2 u_{yyy})].$$

According to (10), the discrete approximation given in (6) and (7) has local truncation errors of order $O(h^4)$. On dividing by $h^2$, we see that the differential operator is approximated to a local accuracy of order $O(h^2)$. Even if a more accurate quadrature scheme is used for integrating (3), $O(h^4)$ truncation errors remain. Their precise forms depend on the integration rule used. It follows that the estimate in (8) gives the best possible rate for the nodal solution error irrespective of the increase in quadrature accuracy.

This conclusion also holds for the case of smooth variable coefficients, since their variations only introduce $O(h^4)$ truncation errors. (See the Appendix for an example.)

**2.2. Derivative superconvergence points.** Consider first the problem of derivative calculation from the bilinear finite-element nodal interpolant $u_I$ of $u$. Simply differentiating the expansion on element $\Omega_e$, we have

(12) $$u_{Ix}(\bar{x}, \bar{y}) = \sum_{j=1}^{4} u_j \psi_{jx}(\bar{x}, \bar{y})$$

where $u_j$ are the interpolated nodal values for $\Omega_e$, $\psi_j$ are the element basis functions, and $\bar{x}, \bar{y}$ is an arbitrary point in the element.

Next, we introduce Taylor series expansions for $u_j = u(x_j, y_j)$ about $x = \bar{x}$, $y = \bar{y}$ with $\delta_j^x = x_j - \bar{x}$, $\delta_j^y = y_j - \bar{y}$ to obtain

(13) $$\begin{aligned} u_j = u(x_j, y_j) &= u(\bar{x}, \bar{y}) + u_x(\bar{x}, \bar{y})\delta_j^x + u_y(\bar{x}, \bar{y})\delta_j^y \\ &+ u_{xx}(\bar{x}, \bar{y})(\delta_j^x)^2/2! + u_{yy}(\bar{x}, \bar{y})(\delta_j^y)^2/2! + u_{xy}(\bar{x}, \bar{y})\delta_j^x\delta_j^y + \cdots. \end{aligned}$$

Using (13) in the right side of (12) and regrouping terms, we have

(14) $$\begin{aligned} \bar{u}_x = \sum_{j=1}^{4} \bar{\psi}_{jx} u_j - \left[\frac{1}{2!}\sum_{j=1}^{4} \bar{\psi}_{jx}(\delta_j^x)^2\right]\bar{u}_{xx} \\ - \left[\frac{1}{2!}\sum_{j=1}^{4} \bar{\psi}_{jx}(\delta_j^y)^2\right]\bar{u}_{yy} - \left[\sum_{j=1}^{4} \bar{\psi}_{jx}\delta_j^x\delta_j^y\right]\bar{u}_{xy} + O(h^2) \end{aligned}$$

where for notational convenience $\bar{u} = u(\bar{x}, \bar{y})$, $\bar{u}_x = u_x(\bar{x}, \bar{y})$, $\bar{\psi}_{jx} = \psi_{jx}(\bar{x}, \bar{y})$, and so on. A similar expression holds for $\bar{u}_y$.

Now the derivative of the approximate solution $u_h$ at $\bar{x}, \bar{y}$ in $\Omega_e$ is

(15) $$\bar{u}_{hx} = \sum_{j=1}^{4} u_{hj}\bar{\psi}_{jx}.$$

Subtracting (15) from (14) yields the error in the derivative

(16) $$\begin{aligned} \bar{e}_x = \sum_{j=1}^{4} e_j\bar{\psi}_{jx} - \left[\frac{1}{2!}\sum_{j=1}^{4} \bar{\psi}_{jx}(\delta_j^x)^2\right]\bar{u}_{xx} \\ - \left[\frac{1}{2!}\sum_{j=1}^{4} \bar{\psi}_{jx}(\delta_j^y)^2\right]\bar{u}_{yy} - \left[\sum_{j=1}^{4} \bar{\psi}_{jx}\delta_j^x\delta_j^y\right]\bar{u}_{xy} + O(h^2). \end{aligned}$$

When we introduce nodal estimate (8) for $e_j$ and use the fact that derivatives and hence difference quotients of $\phi$ are bounded, the first term on the right in (16) satisfies

$$\left|\sum_{j=1}^{4} \bar{\psi}_{jx} e_j\right| = O(h^2).$$

This implies that (16) will be an $O(h^2)$ approximation, provided the remaining first-order terms are zero or collectively cancel. On examination, we find that coefficients of $\bar{u}_{xy}$ and $\bar{u}_{yy}$ are zero for all $\bar{x}$, $\bar{y}$ in $\Omega_e$, but the coefficient of $\bar{u}_{xx}$ is zero for all $\bar{y}$ with $\bar{x} = (x_1 + x_2)/2$. Therefore, $u_{hx}$ is superconvergent on the *line* bisecting the horizontal sides of $\Omega_e$. Similarly, $u_{hy}$ is $O(h^2)$ along the line bisecting vertical sides of $\Omega_e$. Hence, the centroid (Gauss point) is the superconvergent point for both $u_{hx}$ and $u_{hy}$.

**2.3. Nodal derivative extraction.** Now consider the calculation of derivatives (flux components, stresses) at interior node point $x_i, y_i$. For a solution $u$ of (1) having sufficient smoothness in the interior of $\Omega$, Bramble and Hubbard [1] prove the following estimate for the equivalent finite-difference scheme:

$$|D_h^n e(x_i, y_i)| \leqq c_n[|e|_{\Omega_h} + O(h^2)] \tag{17}$$

where $D_h^n$ is an $n$th order difference quotient having $O(h^2)$ truncation error, $c_n$ is a constant independent of $h$, and $|e|_{\Omega_h} = \max_{x_i, y_i} |e(x_i, y_i)|$. For the problem considered here we have, according to (8),

$$|e|_{\Omega_h} = O(h^2).$$

Thus, (17) becomes

$$|D_h^n e(x_i, y_i)| \leqq Ch^2. \tag{18}$$

This result can now be used to derive a superconvergent approximation for the flux components $a_1 u_x$, $a_2 u_y$ (and hence derivatives if desired) at node point $x_i$, $y_i$.

A Taylor series expansion for $u(x_i - h, y_i)$ about $(x_i, y_i)$ yields

$$a_1 u_x(x_i, y_i) = \frac{a_1}{h}[u(x_i, y_i) - u(x_i - h, y_i)] + \frac{h}{2} a_1 u_{xx}(x_i, y_i) + O(h^2). \tag{19}$$

Replacing $a_1 u_{xx}(x_i, y_i)$ in (19) using differential equation (1) and then introducing the following difference formulas for $a_{1x}$, $u_x$, $u_y$, and $(a_2 u_y)_y$

$$a_{1x}(x_i, y_i) = \frac{a_1(x_i, y_i) - a_1(x_i - h, y_i)}{h} + O(h),$$

$$u_x(x_i, y_i) = \frac{u(x_i, y_i) - u(x_i - h, y_i)}{h} + O(h),$$

$$u_y(x_i, y_i) = \frac{u(x_i, y_i + h) - u(x_i, y_i - h)}{2h} + O(h^2),$$

$$(a_2 u_y(x_i, y_i))_y = \frac{(a_2(x_i, y_i + h) + a_2(x_i, y_i))}{2h^2}[u(x_i, y_i + h) - u(x_i, y_i)]$$
$$- \frac{(a_2(x_i, y_i) + a_2(x_i, y_i - h))}{2h^2}[u(x_i, y_i) - u(x_i, y_i - h)] + O(h^2),$$

we obtain

$$
\begin{aligned}
a_1 u_x(x_i, y_i) = {} & \frac{(a_1(x_i, y_i) + a_1(x_i - h, y_i))}{2h}[u(x_i, y_i) - u(x_i - h, y_i)] \\
& + \frac{(a_2(x_i, y_i + h) + a_2(x_i, y_i))}{4h}[u(x_i, y_i + h) - u(x_i, y_i)] \\
& - \frac{(a_2(x_i, y_i) + a_2(x_i, y_i - h))}{4h}[u(x_i, y_i) - u(x_i, y_i - h)] \\
& + \frac{b_1}{2}[u(x_i, y_i) - u(x_i - h, y_i)] + \frac{b_2}{4}[u(x_i, y_i + h) - u(x_i, y_i - h)] \\
& + c\frac{h}{2}u(x_i, y_i) - \frac{h}{2}f(x_i, y_i) + O(h^2).
\end{aligned}
\tag{20}
$$

Note that (20) is an $O(h^2)$ difference formula involving nodal values of the exact solution $u$. On introducing the finite-element approximation $u_h$ for $u$ on the right in (20), we define the approximation for $a_1 u_x(x_i, y_i)$

$$
\begin{aligned}
a_1 u_x^*(x_i, y_i) = {} & \frac{(a_1(x_i, y_i) + a_1(x_i - h, y_i))}{2h}[u_h(x_i, y_i) - u_h(x_i - h, y_i)] \\
& + \frac{(a_2(x_i, y_i + h) + a_2(x_i, y_i))}{4h}[u_h(x_i, y_i + h) - u_h(x_i, y_i)] \\
& - \frac{(a_2(x_i, y_i) + a_2(x_i, y_i - h))}{4h}[u_h(x_i, y_i) - u_h(x_i, y_i - h)] \\
& + \frac{b_1}{2}[u_h(x_i, y_i) - u_h(x_i - h, y_i)] + \frac{b_2}{4}[u_h(x_i, y_i + h) - u_h(x_i, y_i - h)] \\
& + \frac{ch}{2}u_h(x_i, y_i) - \frac{h}{2}f(x_i, y_i).
\end{aligned}
\tag{21}
$$

Subtracting (21) from (20) and using (18), we find that (21) is a superconvergent $O(h^2)$ flux approximation. (For a related study of derivative approximations see MacKinnon and Carey [5].)

Finally, let us use this result to analyze a finite-element projection technique for flux post-processing. This technique is based on the integration-by-parts procedure in the finite-element integral statement, from which we define the projection relationship for $a_1 u_x^*$:

$$
\int_{s_l} a_1 u_x^* v_{hl}\, ds = \int_{\Omega_p} (a_1 u_{hx} v_{hlx} + a_2 u_{hy} v_{hly} + (\mathbf{b} \cdot \nabla u_h + c u_h - f) v_{hl})\, dx\, dy
\tag{22}
$$

where $s_l$ is defined by element sides connecting gridpoints $(x_i, y_i - h)$, $(x_i, y_i)$, $(x_i, y_i + h)$ and $\Omega_p$ represents the two-element patch defined by gridpoints $(x_i - h, y_i)$, $(x_i, y_i)$, $(x_i, y_i + h)$, $(x_i - h, y_i + h)$, $(x_i, y_i - h)$, $(x_i - h, y_i - h)$. This approach has been examined in one dimension by Wheeler [6], Dupont [4], and Carey [2]. In two-dimensional numerical test cases the method has been demonstrated to yield an $O(h^2)$ approximation to the nodal flux when $a_1 u_x^*$ is assumed to be piecewise-constant over $s_l$ (Carey, Chow, and Seager [3]). Indeed, if we integrate (21) using the trapezoidal rule, as described in the Appendix, then the resulting discrete formula for $a_1 u_x^*$ is identical to (21). This then confirms the observed numerical convergence rate of $O(h^2)$. If a more accurate quadrature scheme is used to evaluate (22), the resulting difference formula is, in general, different from (21). However, a simple Taylor series analysis confirms that the formula is $O(h^2)$ accurate.

**3. Nodal solution enhancement.** In this section we apply the results obtained in the foregoing analysis and formulate a new scheme to compute an accurate approximation $e_i^*$ to the gridpoint error $e_i = u(x_i, y_i) - u_h(x_i, y_i)$. This approximation may then be used to improve $u_h$ and, moreover, increase the asymptotic rate of convergence of $u_h$ and its derivatives. We point out that although the formulation presented here is for problems in two-dimensions, it includes the one-dimensional case by simply setting $y$-derivatives equal to zero.

In the interest of clarity, we restrict our analysis to the constant coefficient case described by (6)–(11). The extension to other cases involving different quadrature schemes and variable coefficients is straightforward in view of our previous results.

First recall (10) and (11):

$$B_h(u, v_{hi}) = h^2 Lu(x_i, y_i) + h^4 \tau_i(u) + O(h^6), \tag{10}$$

$$\tau(u) = -\tfrac{1}{12}[a_1 u_{xxxx} + a_2 u_{yyyy} - 2(b_1 u_{xxx} + b_2 u_{yyy})]. \tag{11}$$

Now from estimate (18), since $u_h$ is $O(h^2)$ accurate at the node points, any $n$th-order difference quotient $D_h^n$ of $u_h$ also converges to the exact value $D^n u$ at a rate of $O(h^2)$. Therefore, at any interior node point $i$, $\tau(u)$ can be rewritten using difference quotients $D_h^n u_h$ as

$$\begin{aligned} \tau_i(u) &= -\tfrac{1}{12}[a_1 D_{hxi}^4 u_h + a_2 D_{hyi}^4 u_h - 2(b_1 D_{hxi}^3 u_h + b_2 D_{hyi}^3 u_h)] + O(h^2) \\ &\equiv \tau_{hi}(u_h) + O(h^2). \end{aligned} \tag{23}$$

(Note that since the fifth derivatives of $u$ are assumed bounded, then $\tau_i(u)$ at node points on boundary $\partial\Omega$ can also be approximated to $O(h^2)$ accuracy by simply using an $O(h^2)$ extrapolation to the boundary.)

Next, interpolate the nodal values $\tau_i(u)$ in the piecewise-bilinear basis as

$$\tau(u) = \sum_{j=1}^{N} \tau_j(u)\psi_j(x, y) + O(h^2) \tag{24}$$

where $N$ is the number of node points.

Introducing (23) in (24), we have

$$\tau(u) = \sum_{j=1}^{N} \tau_{hj}(u_h)\psi_j(x, y) + O(h^2). \tag{25}$$

Replacing $\tau_i(u)$ in (10) using (25), we have

$$B_h(u, v_{hi}) = h^2 Lu(x_i, y_i) + h^4 \tau_{hi}(u_h) + O(h^6). \tag{26}$$

Using (26) in place of (10), the estimate (8) now has the form

$$e_i = \phi^*(x_i, y_i)h^2 + R^*(x_i, y_i, h)h^3 \tag{27}$$

where $\phi^*$ satisfies the auxiliary problem

$$\begin{aligned} L\phi^* = \tau_h(u_h) &= \sum_{j=1}^{N} \tau_{hj}(u_h)\psi_j(x, y) \quad \text{in } \Omega, \\ \phi^* &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{28}$$

The objective now is to construct a finite-element approximation $\phi_h^*$ to $\phi^*$ in (28), and then use this approximation in the leading term on the right of (28) to obtain an accurate correction to the nodal solution.

First, let us assume that we have already computed $u_h$ from (5) using $LU$ factorization and have saved the computed matrix factors. The Galerkin finite-element approximation to (28) is as follows. Find $\phi_h^* \in H^h$ such that

$$B(\phi_h^*, v_h) = (\tau_h(u_h), v_h) \tag{29}$$

for all $v_h \in H^h \subset H_0^1$. As before, if we evaluate integrals in (29) using the $(2\times 2)$ trapezoidal integration rule we get

$$\mathbf{B}_h \boldsymbol{\phi}_h^* = \boldsymbol{\tau}_h(u_h) h^2 \tag{30}$$

where $\tau_{hi}(u_h)$ is defined in (23) and $B_h(\phi_h^*, v_{hi})$ is analogous to (6).

Since the matrix factorization of $\mathbf{B}_h$ is already given from the previous calculation of $\mathbf{u}_h$, the approximate function $\phi_h^*$ in (30) can be computed efficiently once $\tau_{hi}(u_h)$ are computed (for the Dirichlet problem $\tau_{hi}(u_h)$ is needed at interior points only); $\tau_{hi}(u_h)$ is easily computed using one-dimensional difference formulas. For example, we may write

$$\begin{aligned} D_\xi^n u &= \sum_{j=1}^{k+1} \frac{d^n}{d\xi^n} \Psi_j(\xi) u(\xi_j) + O(h^{k+1-n}) \\ &= D_{h\xi}^n u + O(h^{k+1-n}), \qquad k \geqq n, \quad \xi = x, y \end{aligned} \tag{31}$$

where $u(\xi_j)$ are node point values of function $u$; and $\Psi_j$ are Lagrange polynomial shape functions of degree $k$. In particular, for a second-order $(O(h^2))$ approximation to $d^4u/dx^4$, $n=4$ and $k=5$. Note that (31) can be used to approximate $d^4u/dx^4$ at interior nodes near the boundary. For this case (31) is simply a one-sided difference formula involving interior node point values of $u$ only.

Solution $\phi_h^*$ from (30) will approximate $\phi^*$ with accuracy $O(h^p)$ at all node points, where $p$ depends on the smoothness of solution $u$ to (1). Note that $\tau$ in (24) is $C^1$ in view of the assumptions on $u$. Moreover, $\tau_h$ in (28) is $C^0$ by construction so $\phi^* \in C^2$ and $p \geqq 1$. Replacing $\phi^*$ in (28) with $\phi_h^*$, we have

$$\begin{aligned} e_i^* &= \phi_h^*(x_i, y_i) h^2 + R^*(x_i, y_i, h) h^3 + O(h^{2+p}) \\ &= \phi_{hi}^* h^2 + O(h^{2+p}) \end{aligned} \tag{32}$$

since $R^*$ is $O(h)$.

This important result implies that we can compute node point errors $e_i^*$ having at least $O(h^3)$ accuracy, and $O(h^4)$ accuracy $(p=2)$ for sufficiently smooth solution. An immediate consequence of this result is that we can also increase the accuracy of our approximation $u_h$ (and its derivatives, if desired) from $O(h^2)$ to at least $O(h^3)$. That is, the enhanced gridpoint value obtained by adding the nodal correction $e_i^*$ becomes

$$u_{hi}^* = u_{hi} + e_i^*. \tag{33}$$

The solution enhancement procedure may be summarized as follows:

(1) Solve the finite-element problem $\mathbf{B}_h \mathbf{u}_h = \mathbf{f}_h$ using sparse $LU$ factorization and save matrix factors.

(2) "Process" approximation $\mathbf{u}_h$ and form associated vector $\boldsymbol{\tau}_h(u_h)$. Then, using matrix factors of $\mathbf{B}_h$, solve auxiliary problem $\mathbf{B}_h \boldsymbol{\phi}_h^* = \boldsymbol{\tau}_h(u_h)$.

(3) Compute approximate node point error correction

$$e_i^* = \phi_h^*(x_i, y_i)h^2$$

and hence the "enhanced" solution

$$u_i^* = u_{hi} + e_i^*.$$

**3.1. Numerical examples.** Numerical test studies have been made to demonstrate the effectiveness of the nodal enhancement post-processing procedure. (For results related to the application of post-processing derivative formula (22) and related formulas, we refer the reader to Carey [2], Carey, Chow, and Seager [3], and MacKinnon and Carey [5].)

In the first test case we consider the two-point boundary value problem

$$-u_{xx} + u_x + u = f, \qquad 0 < x < 1, \tag{34}$$
$$u(0) = u(1) = 0$$

where $f$ is constructed such that the analytic solution is $u = x(1-x)(1+x)^5$.

We take a sequence of uniform mesh refinements with $h = \frac{1}{5}, \frac{1}{10}, \frac{1}{20}$, and $\frac{1}{40}$. Numerical integration is performed using the trapezoidal rule, and derivatives $u_{xxx}$ and $u_{xxxx}$ in $\tau$ are approximated to order $O(h^3)$ and $O(h^2)$, respectively, by six-point difference formulas. A six-point formula for $u_{xxx}$ was used because it is computationally convenient to simply differentiate this formula and use the result to approximate $u_{xxxx}$.

Node point errors $E_i$, $E_i^*$ for approximations $u_{hi}$ and enhancement $u_i^*$ are presented in Table 1. Note the substantial increase in accuracy and asymptotic rates of convergence afforded by the enhancement procedure.

Next we examine three approximations to $u_x$ at $x = 1$. These approximations are: the standard $O(h)$ derivative approximation $u_{hx}$; the post-processed derivative $u_x^*$ given by (21); and the enhanced derivative denoted by $u_x^{**}$ and given by an $O(h^4)$ six-point difference formula operating on enhanced solution $u^*$. Results are presented in Table 2. Approximations $u_x^*$ and $u_x^{**}$ are $O(h^2)$ and $O(h^4)$ accurate as predicted.

TABLE 1
*Node point errors $E(x_i)$, $E^*(x_i)$ for the case $b = 1$.*

| $h$ | $E(0.2)$ | $E^*(0.2)$ | $E(0.4)$ | $E^*(0.4)$ | $E(0.6)$ | $E^*(0.6)$ | $E(0.8)$ | $E^*(0.8)$ |
|---|---|---|---|---|---|---|---|---|
| $\frac{1}{5}$ | 0.085004 | 0.016524 | 0.161265 | 0.010216 | 0.200329 | 0.009952 | 0.162612 | 0.015906 |
| $\frac{1}{10}$ | 0.021283 | 0.204E−3 | 0.040342 | 0.598E−3 | 0.050081 | 0.724E−3 | 0.040634 | 0.389E−3 |
| $\frac{1}{20}$ | 0.005323 | 0.27E−4 | 0.010087 | 0.51E−4 | 0.012520 | 0.61E−4 | 0.010157 | 0.46E−4 |
| $\frac{1}{40}$ | 0.001330 | 0.18E−5 | 0.002521 | 0.33E−5 | 0.0031301 | 0.40E−5 | 0.002539 | 0.31E−5 |
| | $\sim O(h^2)$ | $\sim O(h^4)$ | $\sim O(h^2)$ | $\sim O(h^4)$ | $\sim O(h^2)$ | $\sim O(h^4)$ | $\sim O(h^2)$ | $\sim O(h^4)$ |

TABLE 2
*Derivatives $u_{hx}$, $u_x^*$, and $u_x^{**}$ at $x = 1$. The exact derivative is $u_x(1) = -32.0$.*

| $h$ | $u_{hx}$ | $u_x^*$ | $u_x^{**}$ | $\lvert u_x - u_{hx}\rvert$ | $\lvert u_x - u_x^*\rvert$ | $\lvert u_x - u_x^{**}\rvert$ |
|---|---|---|---|---|---|---|
| $\frac{1}{5}$ | −14.303480 | −34.933828 | −31.476970 | 17.69652 | 2.933828 | 0.5230300 |
| $\frac{1}{10}$ | −22.034701 | −32.736437 | −31.982434 | 9.965299 | 0.7364370 | 0.0175660 |
| $\frac{1}{20}$ | −26.716385 | −32.184295 | −31.999583 | 5.283615 | 0.1842950 | 0.417E−3 |
| $\frac{1}{40}$ | −29.280084 | −32.046085 | −31.999975 | 2.719916 | 0.0460850 | 0.25E−4 |
| | | | | $\sim O(h)$ | $\sim O(h^2)$ | $\sim O(h^4)$ |

As a two-dimensional test problem we take the example

$$(u_{xx}+u_{yy})=(2-42x^5)(y-y^7)-42y^5(x^2-x^7) \quad \text{in } \Omega=(0,1)\times(0,1)$$

with

$$u=0 \quad \text{on } \partial\Omega. \tag{35}$$

The analytic solution is the polynomial

$$u=(x^2-x^7)(y-y^7). \tag{36}$$

Node point results for a sequence of calculations on uniformly refined meshes of $h=\frac{1}{5},\frac{1}{10}$, and $\frac{1}{30}$ are given in Tables 3–5. Again, the observed rates of convergence corroborate our analysis.

**Conclusion.** By a suitable choice of quadrature rule the finite-element approximation for a two-dimensional elliptic problem has been related to a familiar finite-difference approximation. Nodal superconvergence of the solution then follows from an estimate of finite-difference theory. Moreover, any $n$th order difference approximation having Taylor series truncation error of $O(h^2)$ at a node point converges to the exact value at a rate of $O(h^2)$. Therefore, accurate derivative extraction formulas can be derived directly using Taylor series ideas.

TABLE 3
*Node point errors $E(x_i)$, $E^*(x_i)$ along $x=0.8$.*

| $h$ | $E(0.8, 0.2)$ | $E^*(0.8, 0.2)$ | $E(0.8, 0.4)$ | $E^*(0.8, 0.4)$ | $E(0.8, 0.6)$ | $E^*(0.8, 0.6)$ | $E(0.8, 0.8)$ | $E^*(0.8, 0.8)$ |
|---|---|---|---|---|---|---|---|---|
| $\frac{1}{5}$ | 0.012138 | 0.004074 | 0.023837 | 0.005127 | 0.032477 | 0.006333 | 0.030038 | 0.006824 |
| $\frac{1}{10}$ | 0.003102 | 0.40E−4 | 0.006087 | 0.60E−4 | 0.008275 | 0.80E−4 | 0.007625 | 0.93E−4 |
| $\frac{1}{30}$ | 0.346E−3 | 0.2E−6 | 0.680E−3 | 0.3E−6 | 0.924E−3 | 0.4E−6 | 0.850E−3 | 0.4E−6 |
| | $\sim O(h^2)$ | $\sim O(h^4)$ | $\sim O(h^2)$ | $\sim O(h^4)$ | $\sim O(h^2)$ | $\sim O(h^4)$ | $\sim O(h^2)$ | $\sim O(h^4)$ |

TABLE 4
*Node point errors $E(x_i)$, $E^*(x_i)$ along $y=0.8$.*

| $h$ | $E(0.2, 0.8)$ | $E^*(0.2, 0.8)$ | $E(0.4, 0.8)$ | $E^*(0.4, 0.8)$ | $E(0.6, 0.8)$ | $E^*(0.6, 0.8)$ |
|---|---|---|---|---|---|---|
| $\frac{1}{5}$ | 0.007514 | 0.003668 | 0.017644 | 0.003680 | 0.028428 | 0.005004 |
| $\frac{1}{10}$ | 0.001982 | 0.21E−4 | 0.004496 | 0.22E−4 | 0.007233 | 0.48E−4 |
| $\frac{1}{30}$ | 0.214E−3 | 0.2E−6 | 0.502E−3 | 0.4E−6 | 0.807E−3 | 0.5E−6 |
| | $\sim O(h^2)$ | $\sim O(h^4)$ | $\sim O(h^2)$ | $\sim O(h^4)$ | $\sim O(h^2)$ | $\sim O(h^4)$ |

TABLE 5
*Derivatives $u_{hx}$, $u_x^*$, and $u_x^{**}$ at $(x, y)=(0.8, 0.8)$. The exact derivative is $u_x(0.8, 0.8)=-0.1387215$.*

| $h$ | $u_{hx}$ | $u_x^*$ | $u_x^{**}$ | $\|u_x-u_{hx}\|$ | $\|u_x-u_x^*\|$ | $\|u_x-u_x^{**}\|$ |
|---|---|---|---|---|---|---|
| $\frac{1}{5}$ | 0.282010 | −0.405757 | −0.155165 | 0.420731 | 0.267035 | 0.016443 |
| $\frac{1}{10}$ | 0.137307 | −0.208948 | −0.139683 | 0.276028 | 0.070227 | 0.961E−3 |
| $\frac{1}{30}$ | −0.0310355 | −0.146723 | −0.138726 | 0.107686 | 0.008001 | 0.4E−5 |
| | | | | $\sim O(h)$ | $\sim O(h^2)$ | $\sim O(h^4)$ |

We emphasize that since this Taylor series approach relies only on elementary analysis concepts, it is straightforward to understand and implement. Furthermore, although this is not taken up here, the method can be easily applied to higher-order elements and problems in three dimensions. Also, derivatives can be extracted from finite-difference solutions in the same manner.

Finally, using the truncation error in an auxiliary problem the nodal superconvergence property can be further exploited to enhance the gridpoint solution accuracy. These results are of practical significance in solution and derivative post-processing and also for a posteriori error analysis in conjunction with adaptive refinement. The adaptive refinement aspects will be taken up in future studies.

**Appendix. Trapezoidal rule and variable coefficients.** For the case of variable coefficients and trapezoidal integration, we have from (3) at interior gridpoint $i$

$$B_h(u_h, v_{hi}) = f_{hi} \tag{1.1}$$

where

$$\begin{aligned} B_h(u_h, v_{hi}) = -\Bigg\{ & \frac{a_1(x_i+h, y_i)+a_1(x_i, y_i)}{2h}[u_h(x_i+h, y_i)-u_h(x_i, y_i)] \\ & -\frac{a_1(x_i, y_i)+a_1(x_i-h, y_i)}{2h}[u_h(x_i, y_i)-u_h(x_i-h, y_i)] \\ & +\frac{a_2(x_i, y_i+h)+a_2(x_i, y_i)}{2h}[u_h(x_i, y_i+h)-u_h(x_i, y_i)] \\ & -\frac{a_2(x_i, y_i)+a_2(x_i, y_i-h)}{2h}[u_h(x_i, y_i)-u_h(x_i, y_i-h)]\Bigg\} \\ & +h\frac{b_1}{2}(x_i, y_i)[u_h(x_i+h, y_i)-u_h(x_i-h, y_i)] \\ & +h\frac{b_2}{2}(x_i, y_i)[u_h(x_i, y_i+h)-u_h(x_i, y_i-h)]+c(x_i, y_i)u_h(x_i, y_i)h^2 \end{aligned} \tag{1.2}$$

and

$$f_{hi} = f(x_i, y_i)h^2. \tag{1.3}$$

For a smooth function $w$, and using Taylor's theorem,

$$B_h(w, v_{hi}) = h^2 Lw(x_i, y_i) + h^4\tau_i(w) + O(h^6) \tag{1.4}$$

where

$$\begin{aligned} \tau(w) = -\tfrac{1}{12}[ & a_1 w_{xxxx} + a_{1x}w_{xxx} + a_{1xx}w_{xx} + a_{1xxx}w_x \\ & + a_2 w_{yyyy} + a_{2y}w_{yyy} + a_{2yy}w_{yy} + a_{2yyy}w_y - 2(b_1 w_{xxx} + b_2 w_{yyy})]. \end{aligned} \tag{1.5}$$

From (1.4) and (1.5) we see that variable coefficients produce additional $O(h^4)$ terms in $\tau$. Hence, the accuracy of (1.1) remains $O(h^2)$ and the results demonstrated for the constant coefficient case extend directly to the case of variable coefficients.

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# Natural and Postprocessed Superconvergence in Semilinear Problems

**S.-S. Chow,[1] G. F. Carey,[2] and R. D. Lazarov[3]**
*[1]University of Wyoming, Laramie, Wyoming [2]University of Texas at Austin, Austin, Texas 78712 and [3]Bulgarian Academy of Science, Sofia, Bulgaria*

Superconvergence error estimates are established for a class of semilinear problems defined by a linear elliptic operator with a nonlinear forcing term. The analysis is for rectangular biquadratic elements, and we prove superconvergence of the derivative components along associated lines through the Gauss points. Derivative postprocessing formula and formulas for integrals are also considered and similar superconvergence estimates proven.

## 1. INTRODUCTION

In recent years, much effort has been focused on superconvergence phenomena in the numerical solution of differential equations. In the context of finite element methods, it was observed that under certain conditions the convergence rate at specific points or along specific lines is higher than the global convergence rate of the approximations. These phenomena may occur naturally or "artificially" from postprocessing procedures. For linear problems, an extensive review on superconvergence may be found in the paper by Křížek and Neittaanmäki [1]. For nonlinear problems, few results are available, even though numerical evidence strongly suggests that most of the results for the linear problems may be carried over to reasonably well behaved nonlinear problems. (See Chow and Lazarov [2], Wheeler and Carey [3], and Carey et al. [4].)

In this paper, we identify a class of semilinear problems for which many superconvergence results remain valid. In particular, by restricting ourselves to biquadratic elements for simplicity, we establish superconvergence of approximate derivative components along lines through the Gauss points and superconvergence of the gradient at the Gauss points. We also prove superconvergence of the boundary flux and linear functionals obtained from certain postprocessing procedures.

The standard notational conventions for Sobolev and Hilbert spaces are employed (e.g., see Showalter [15]). For a given domain $\Omega$, the Sobolev

*Please address correspondence to Dr. G. F. Carey, Department of Aerospace Engineering and Engineering Mechanics, University of Texas at Austin, Austin, TX 78712-1085.

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space $W^{k,p}(\Omega)$, $k = \{0, \pm 1, \pm 2, \ldots\}$, $p \in [1, \infty]$, is equipped with the usual norm $\|\cdot\|_{k,p,\Omega}$ and seminorm $|\cdot|_{k,p,\Omega}$. We shall omit the index $p$ when $p = 2$, and write $H^k$ for $W^{k,2}$ and $H$ for $H_0^1(\Omega)$, the subspace of $H^1(\Omega)$ with elements vanishing on the boundary of $\Omega$. Since the distinction should be clear from the context, we use $(\cdot,\cdot)$ to denote both the $L^2(\Omega)$ inner product and the $H^{-1}(\Omega) \times H_0^1(\Omega)$ duality pairing.

For the present superconvergence analysis, we restrict the domain $\Omega$ throughout to be a subdomain of $\mathbb{R}^2$ consisting of a union of rectangles with sides parallel to the coordinate axes and let $\partial\Omega$ denote the boundary. We specifically consider the class of semilinear boundary value problems defined by

$$Lu \equiv -\frac{\partial}{\partial x}\left(\alpha(x,y)\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(\beta(x,y)\frac{\partial u}{\partial y}\right) = f(u), \quad \text{in } \Omega \qquad (1)$$
$$u = 0, \quad \text{on } \partial\Omega$$

where (i) $\alpha, \beta$ are uniformly Lipschitz continuous in $\Omega$; (ii) $\alpha, \beta$ are uniformly bounded above and below by some positive constants $c_1$ and $c_0$, respectively; (iii) $f(u)$ is a monotonically decreasing Lipschitz continuous function of $u \in \mathbb{R}$ such that for all $v, w$ in $H$,

$$[f(w) - f(v), w - v] \geq \gamma\|w - v\|_0^2 \qquad (2)$$

with $\gamma > 0$; and (iv) for all $v, w$ in $H$, there exists a constant $\delta > 0$ such that

$$c_0|w - v|_1^2 - \gamma\|w - v\|_0^2 \geq \delta|w - v|_1^2. \qquad (3)$$

Using the Green-Gauss identities, the corresponding weak statement of eqs. (1) and (2) becomes: Find $u \in H$ such that

$$a(u,v) = [f(u),v] \quad \text{for all } v \in H, \qquad (4)$$

where $a(u,v) \equiv \iint_\Omega(\alpha\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \beta\frac{\partial u}{\partial y}\frac{\partial v}{\partial y})\,dx\,dy$. Under the assumptions (i)–(iv) following eq. (1), it is easy to verify that the energy functional associated with $a(u,v)$ is strictly convex, and thus the existence and uniqueness of the weak solution to the variational problem (4) is guaranteed.

To construct the finite element spaces, we first partition $\Omega$ with rectangular elements whose sides are parallel to the coordinate axes. We assume that this discretization $T_h$ is regular [6], with the parameter $h$ denoting the maximum diameter of the elements in $T_h$, and satisfies the inverse hypothesis $h_{\min} \geq Ch$ (throughout this paper, $C$ is taken as a generic constant and is independent of $h$). Introducing a local polynomial basis of degree $k$ on each element and imposing continuity across element boundaries, we define the standard $C^0$ Lagrange piecewise-polynomial finite element space $S^h \subset H^1(\Omega)$. Let $S_0^h$ denote the subspace of functions in $S^h$ that vanish on $\partial\Omega$. The finite element approximation $u_h \in S_0^h$ of eq. (4) is obtained by solving

$$a(u_h,v_h) = (f(u_h),v_h) \quad \text{for all } v_h \text{ in } S_0^h. \qquad (5)$$

Using the idea of an elliptic projection operator $P$ on $H$ defined by

$$a(u - Pu, v_h) = 0 \quad \text{for all } v_h \text{ in } S_0^h, \tag{6}$$

Noor and Whiteman [7] showed that the global energy error estimates for the finite element approximation of degree $k$ defined by eq. (5) are similar to those obtained from standard finite element theory for linear problems, i.e.,

$$\|u - u_h\|_1 \le Ch^k\|u\|_{k+1}. \tag{7}$$

These global estimates are optimal. However, for linear problems, it has been observed in practice that the finite element solutions may achieve a higher order of accuracy $O(h^{k+1})$ in the gradient at certain Gauss points. For linear problems in one dimension this problem, as well as nodal convergence of the solution, has been extensively investigated. In two dimensions this property for the derivative was shown to hold by Oganesjan and Ruhovec [8] and Andreev and Lazarov [9] for linear and quadratic triangular elements, respectively, and by Zlámal [10, 11] and Lesaint and Zlámal [12] for quadrilateral elements.

In numerical studies by Ewing and Wheeler [13] and Ewing et al. [14], it was observed that the respective component derivatives exhibit superconvergence behavior along Gaussian lines. MacKinnon and Carey [15] used a Taylor-series analysis to prove this result and further superconvergence properties. Superconvergence along Gauss lines has also recently been demonstrated by Ewing et al. [16] for the mixed method. It is of interest to see if these naturally occurring superconvergence phenomena are also present for the semilinear problems (1)–(2). Chow and Lazarov [2] and Wheeler and Carey [3] have performed related studies for the one-dimensional nonlinear problem. Here, we consider the two-dimensional case and biquadratic elements ($k = 2$).

Appropriate postprocessing techniques can also be introduced to yield superconvergent results. For example, after the finite element approximation is obtained, an approximate flux function may be computed from this solution using the integration by parts formula (e.g., see Carey et al. [4]). More specifically, recall that the true normal flux $q$ is defined on $\partial\Omega$ by

$$q = \alpha\frac{\partial u}{\partial x}n_1 + \beta\frac{\partial u}{\partial y}n_2,$$

where $(n_1, n_2)^T$ is the outward normal on $\partial\Omega$, and, by virtue of the Green-Gauss identities, that $q$ satisfies the equation

$$\langle q,v\rangle = a(u;v) - (f(u),v) \quad \text{for all } v \in H^1(\Omega), \tag{8}$$

where

$$\langle q,v\rangle \equiv \int_{\partial\Omega} qv\,ds.$$

Thus, an approximate flux $q_h$ may be obtained by solving

$$\langle q_h,v_h\rangle = a(u_h,v_h) - (f(u_h),v_h) \quad \text{for all } v_h \in S^h. \tag{9}$$

Note that for convenience we have neglected the effect of numerical integration in eq. (9). The approximate boundary flux thus computed is superconvergent. This result has been proved for linear triangles by Lazarov et al. [17] and will be shown to hold for quadrilateral elements in the next section. A related result is given by Douglas et al. [18]. Averaging derivatives between elements can also produce superconvergent results (see [15]).

Another postprocessing application arises in the evaluation of integrals such as

$$F = F(u) = \int_{\Omega} u\psi\, dx, \tag{10}$$

where $u$ is the weak solution of eq. (4) and $\psi$ is a sufficiently smooth function. An obvious approximation to eq. (10) is then

$$F_h = F(u_h) = \int_{\Omega} u_h\psi\, dx. \tag{11}$$

With the aid of a negative norm estimate, we show that $F_h$ is a superconvergent approximation to $F$. (For related work, see [20].)

## 2. SUPERCONVERGENCE AT GAUSS POINTS AND ALONG GAUSSIAN LINES

As most of the calculations are carried out over the reference element $\hat{e} = [-1,1] \times [-1,1]$, let us first consider the transformation of an element $e$ to $\hat{e}$ and the corresponding change of the bilinear form $a(\cdot,\cdot)$. Suppose $e$ is an element in $T_h$ with center $(x_0, y_0)$. If the side parallel to the $x$ axis is of length $h_e$ and the side parallel to the $y$ axis is of length $k_e$, then a point $(x, y)$ in $e$ is related to corresponding point $(\xi, \eta)$ in $\hat{e}$ via the linear map

$$x = x(\xi,\eta) = x_0 + \frac{1}{2}h_e\xi, \qquad y = y(\xi,\eta) = y_0 + \frac{1}{2}k_e\eta. \tag{12}$$

Any function $v = v(x,y)$ defined on $e$ may then be transformed as a function $\hat{v} = v[x(\xi,\eta), y(\xi,\eta)]$ and analyzed on $\hat{e}$. Due to the restrictions imposed on the discretization of $\Omega$, the Jacobian is constant on each element with $J_e = \frac{1}{4}h_e k_e$. Moreover, since there are no mixed derivative terms in the original differential equation (1), we see that

$$\begin{aligned} a(u,v) &= \sum_e \iint_e \alpha\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \beta\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\, dx\, dy \\ &= \sum_e \left(\frac{k_e}{h_e}\iint_{\hat{e}} \hat{\alpha}\frac{\partial \hat{u}}{\partial \xi}\frac{\partial \hat{v}}{\partial \xi}\, d\xi\, d\eta + \frac{h_e}{k_e}\iint_{\hat{e}} \hat{\beta}\frac{\partial \hat{u}}{\partial \eta}\frac{\partial \hat{v}}{\partial \eta}\, d\xi\, d\eta\right). \end{aligned} \tag{13}$$

Let $G$ denote the set of all images of Gauss points $(\pm\sqrt{3}/3, \pm\sqrt{3}/3)$ through the mapping (12) to the rectangles in the discretization. We now show that, under appropriate regularity assumptions on $u$, superconvergence in the finite element approximation for the components of the gradient occurs along coordinate lines through these image points.

**Lemma 1.** Let $u \in H^4(\Omega)$, then for any $v_h \in S^h$

$$|a(u - u_I, v_h)| \le Ch^3(|u|_3 + |u|_4)|v_h|_1 ; \tag{14}$$

where $u_I$ is the finite element interpolant of $u$ in $S_0^h$.

**Proof.** The proof follows closely that of Zlámal [10]. Note that we have not imposed any boundary condition on $v_h$ and only seminorms appear on the right-hand side.

We see from eq. (13) that we may estimate $a(u - u_I, v_h)$ term by term. By evaluating the coefficient $\hat{\alpha}$ at the centroid and taking this as constant on an element, we may simplify the integral terms. This implies that on setting $\hat{\alpha}^\circ = \hat{\alpha}(0,0)$, we have to estimate

$$\iint_{\hat{e}} \hat{\alpha} \frac{\partial(\hat{u} - \hat{u}_I)}{\partial \xi} \frac{\partial \hat{v}_h}{\partial \xi} d\xi\, d\eta = \iint_{\hat{e}} (\hat{\alpha} - \hat{\alpha}^\circ) \frac{\partial(\hat{u} - \hat{u}_I)}{\partial \xi} \frac{\partial \hat{v}_h}{\partial \xi} d\xi\, d\eta + \hat{\alpha}^\circ L_1(\hat{u}), \tag{15}$$

where

$$L_1(\hat{u}) = \iint_{\hat{e}} \frac{\partial(\hat{u} - \hat{u}_I)}{\partial \xi} \frac{\partial \hat{v}_h}{\partial \xi} d\xi\, d\eta .$$

Now $\alpha$ is uniformly Lipschitz continuous, so $|\hat{\alpha} - \hat{\alpha}^\circ| \le |\hat{\alpha}|_{1,\infty,\hat{e}}$ and hence we may bound the first term on the right-hand side of eq. (15) by $C|\hat{\alpha}|_{1,\infty,\hat{e}}|\hat{u} - \hat{u}_I|_{1,\hat{e}}|\hat{v}_h|_{1,\hat{e}}$. From standard approximation theory, we have $|\hat{u} - \hat{u}_I|_{1,\hat{e}} \le C|\hat{u}|_{3,\hat{e}}$; applying this result and transforming back to the original element $e$, we have the bound

$$\iint_{\hat{e}} (\hat{\alpha} - \hat{\alpha}^\circ) \frac{\partial(\hat{u} - \hat{u}_I)}{\partial \xi} \frac{\partial \hat{v}_h}{\partial \xi} d\xi\, d\eta \le Ch^3|u|_{3,e}|v_h|_{1,e} . \tag{16}$$

To estimate $L_1(\hat{u})$ we observe that for $\hat{v}_h$ fixed, $L_1$ is a bounded linear functional on $\hat{u} \in H^4(\hat{e})$, so $|L_1(\hat{u})| \le C\|\hat{u}\|_{4,\hat{e}}|\hat{v}_h|_{1,\hat{e}}$. Moreover, note that if $\hat{u}$ were a cubic polynomial on $\hat{e}$, then since $\partial\hat{v}_h/\partial\xi$ is a linear polynomial of $\xi$, we would have $L_1(\hat{u}) = 0$. (We only need to check the cases $\hat{u} = \xi^3$ and $\hat{u} = \eta^3$.). Then, according to the Bramble–Hilbert lemma [6], we may replace the norm on $\hat{u}$ by a seminorm to yield

$$|L_1(\hat{u})| \le C|\hat{u}|_{4,\hat{e}}|\hat{v}_h|_{1,\hat{e}} \le Ch^3|u|_{4,e}|v_h|_{1,e} . \tag{17}$$

Combining estimates (16) and (17) in eq. (15),

$$\left| \iint_e \hat{\alpha} \frac{\partial(\hat{u} - \hat{u}_I)}{\partial \xi} \frac{\partial \hat{v}_h}{\partial \xi} \right| \le Ch^3(|u|_{3,e} + |u|_{4,e})|v_h|_{1,e} . \tag{18}$$

In a similar fashion we may obtain the same bound for terms involving $\eta$ derivatives. Thus we have

$$\begin{aligned} |a(u - u_I, v_h)| &\leq \sum_e Ch^3(|u|_{3,e} + |u|_{4,e})|v_h|_{1,e} \\ &\leq Ch^3(|u|_3 + |u|_4)|v_h|_1 \end{aligned}$$

as stated. ∎

From Lemma 1, we may derive various global superconvergence results relating the finite element approximation $u_h$, the interpolant $u_I$ of $u$ in $S_0^h$, and the elliptic projection $Pu$ of $u$ in $S_0^h$.

**Lemma 2.** Let $u \in H^4(\Omega)$ be the solution of eq. (4). Then

$$\|u_I - u_h\|_1 \leq Ch^3\|u\|_4 \tag{19}$$

and

$$\|Pu - u_h\|_1 \leq Ch^3\|u\|_4. \tag{20}$$

**Proof.** From eqs. (4) and (5), we have for all $v_h$ in $S_0^h$

$$|a(u_h - u_I, v_h)| \leq |a(u - u_I, v_h)| + |(f(u_h) - f(u), v_h)|.$$

Using Lemma 1 and the Lipschitz property of $f$ we get

$$\begin{aligned} |a(u_h - u_I, v_h)| &\leq Ch^3(|u|_3 + |u|_4)|v_h|_1 + C\|u - v_h\|_0\|v_h\|_0 \\ &\leq Ch^3\|u\|_4\|v_h\|_1, \end{aligned} \tag{21}$$

where we have assumed the global $L^2$-error estimate $\|u - u_h\|_0 \leq Ch^3\|u\|_3$ for the finite element approximation. This is the standard result for the linear problem and also holds for the semilinear problem, as seen in Section 4 using negative norm estimates.

Finally, noting that $u_h - u_I \in S_0^h$, and that $a(\cdot,\cdot)$ is $H$-elliptic, we set $v_h = u_h - u_I$ to obtain the estimate in eq. (19).

To see that the estimate (20) holds, we first show that $\|Pu - u_I\|_1 \leq Ch^3(|u|_3 + |u|_4)$. Now $a(u - u_I, v_h) = a(Pu - u_I, v_h)$ for all $v_h$ in $S_0^h$, so by Lemma 1 we have $|a(Pu - u_I, v_h)| \leq Ch^3(|u|_3 + |u|_4)|v_h|_1$. Again observing that $Pu - u_I$ is in $S_0^h$, we set $v_h = Pu - u_I$ and make use of the $H$-ellipticity of $a(\cdot,\cdot)$ to obtain the desired bound. The estimate (20) then follows easily by applying the triangle inequality. ∎

The above lemma indicates that both the finite element approximation of $u$ and the elliptic projection of $u$ are exceptionally close to the interpolant of $u$. As we shall see in the next theorems, the derivatives of the interpolant are exceptionally close to those of the solution at Gauss points and along Gauss lines. This can be used to then show the derivative superconvergence property of the finite element approximation.

**Theorem 1.** Let $u \in H^4(\Omega)$, then the arithmetic mean

$$\bar{A} = \frac{1}{N_G} \sum_{g=1}^{N_G} |\nabla u(\mathbf{x}_g) - \nabla u_h(\mathbf{x}_g)|, \tag{22}$$

where $N_G$ denotes the total number of Gauss points $\mathbf{x}_g \in \mathbf{G}$, $g = 1, 2, \ldots, N_G$, is bounded by

$$\bar{A} \leq Ch^3\|u\|_4, \tag{23}$$

or, equivalently,

$$h^2 \sum_{g=1}^{N_G} |\nabla(u - u_h)(\mathbf{x}_g)| \leq Ch^3\|u\|_4. \tag{24}$$

**Proof.** (See also [10].) For Gauss point $(\mathbf{x}_g)$, we have

$$\frac{\partial(u - u_I)}{\partial x} = \frac{2}{h_e} \frac{\partial(\hat{u} - \hat{u}_I)(\zeta)}{\partial \xi},$$

where $\zeta = (\pm\sqrt{3}/3, \pm\sqrt{3}/3)$. Now $L_2(\hat{u}) = (\partial/\partial\xi)(\hat{u} - \hat{u}_I)(\zeta)$ is a linear functional of $\hat{u} \in H^4(\hat{e})$ with $|L_2(\hat{u})| \leq |\hat{u} - \hat{u}_I|_{1,\infty,\hat{e}} \leq C\|\hat{u}\|_{4,\hat{e}}$. In particular, if $\hat{u}$ were a cubic polynomial, then $L_2(\hat{u})$ would vanish identically, which implies on applying the Bramble-Hilbert lemma that

$$|L_2(\hat{u})| \leq C|\hat{u}|_{4,\hat{e}} \leq Ch^3|u|_{4,e}, \tag{25}$$

and

$$\left|\frac{\partial}{\partial x}(u - u_I)(\mathbf{x}_g)\right| \leq Ch^2|u|_{4,e}. \tag{26}$$

A similar bound follows for $(\partial/\partial y)(u - u_I)(\mathbf{x}_g)$, and combining these results

$$|\nabla(u - u_I)(\mathbf{x}_g)| \leq Ch^2|u|_{4,e} \tag{27}$$

at any Gauss point $\mathbf{x}_g \in G$.

Summing over all the Gauss points and taking the average,

$$\begin{aligned} \frac{1}{N_G} \sum_{g=1}^{N_G} |\nabla(u - u_I)(\mathbf{x}_g)| &\leq N_G^{-1}Ch^2 \sum_e |u|_{4,e} \\ &\leq Ch^2|u|_4 N_G^{-1/2} \\ &\leq Ch^3|u|_4, \end{aligned} \tag{28}$$

since $\Sigma_e \iint_{\hat{e}} dx\,dy = \Sigma_e \iint_{\hat{e}} \frac{1}{4} h_e k_e \, d\xi\, d\eta \leq Ch^2 N_G$.

Recall that for $v_h$ belonging to $S^h$, a finite dimensional space, we have the inverse inequality $|v_h|_{0,\infty,e} \leq Ch^{-1}|v_h|_{0,e}$. Setting $v_h = (\partial/\partial x)(u_I - u_h)$ and $v_h = (\partial/\partial y)(u_I - u_h)$ respectively, we can use this inequality to obtain

$$\begin{aligned}\sum_{g=1}^{N_G} |\nabla(u_I - u_h)(\mathbf{x}_g)| &\leq \sum_e |\nabla(u_I - u_h)|_{0,\infty,e} \\ &\leq Ch^{-1} \sum_e |\nabla(u_I - u_h)|_{0,e} \\ &\leq CN_G \|u_I - u_h\|_1 .\end{aligned} \tag{29}$$

Thus applying Lemma 2, we have

$$\frac{1}{N_G} \sum_{g=1}^{N_G} |\nabla(u_I - u_h)(\mathbf{x}_g)| \leq Ch^3 \|u\|_4 . \tag{30}$$

The desired bound on $\bar{A}$ is then obtained by writing $u - u_h = u - u_I + (u_I - u_h)$ in eq. (22) and applying eqs. (28) and (30). The second estimate (24) follows easily from the inverse hypothesis on the triangulation. ∎

Note that for $Pu$ we have a similar result, i.e.,

$$\frac{1}{N_G} \sum_{g=1}^{N_G} |\nabla(u - Pu)(\mathbf{x}_g)| \leq Ch^3(|u|_3 + |u|_4) . \tag{31}$$

This estimate may be obtained by noting that eq. (29) holds with $u_h$ replaced by $Pu$ and that a bound for $\|Pu - u_I\|_1$ is available.

The result in Theorem 1 may be strengthened with the introduction of a discrete $L^2$-norm, as in ref. [11]. The seminorm $|v|_h$ is defined by

$$|v|_h = \left\{ \sum_e \sum_{k=1}^{4} \left[ \frac{k_e}{h_e} \left( \frac{\partial \hat{v}}{\partial \xi}(\zeta_k) \right)^2 + \frac{h_e}{k_e} \left( \frac{\partial \hat{v}}{\partial \eta}(\zeta_k) \right)^2 \right] \right\}^{1/2} , \tag{32}$$

where $\zeta_k = (\pm\sqrt{3}/3, \pm\sqrt{3}/3)$.

This seminorm is equivalent uniformly on $S^h$ to the seminorm $|\cdot|_1$

$$c^{-1}|v_h|_1 \leq |v_h|_h \leq c|v_h|_1 \quad \text{for all } v_h \text{ in } S^h. \tag{33}$$

Since for $v_h \in S_0^h$, the Poincare inequality implies that $|v_h|_1$ is a norm on $S^h$, we see that $|v_h|_h$ is also a norm on $S_0^h$. With this in mind, we write the seminorm $|\cdot|_h$ as $\|\cdot\|_h$.

**Theorem 2.** Let $u \in H^4(\Omega)$, then

$$\|u - u_h\|_h \leq Ch^3 \|u\|_4 \tag{34}$$

and

$$\|u - Pu\|_h \leq Ch^3(|u|_3 + |u|_4) . \tag{35}$$

**Proof.** We first estimate $\|u - u_I\|_h$. From the proof of Theorem 1, we have $|L_2(\hat{u})| \le C|\hat{u}|_{4,\hat{e}}$, and similarly for the $\eta$ derivative term, so

$$\begin{aligned} \|u - u_I\|_h &\le \left(\sum_e C|\hat{u}|^2_{4,\hat{e}}\right)^{1/2} \\ &\le \left(\sum_e Ch^6|u|^2_{4,e}\right)^{1/2} \\ &\le Ch^3|u|_4 . \end{aligned} \tag{36}$$

Next observe that $(u_I - u_h) \in S_0^h$. Applying eq. (33) and Lemma 2, we obtain

$$\|u_I - u_h\|_h \le Ch^3\|u\|_4$$

and

$$\|u_I - Pu\|_h \le Ch^3(|u|_3 + |u|_4). \tag{37}$$

The triangle inequality may now be used to show the desired estimates. This is a stronger result than Theorem 1 since $\bar{A} \le C\|u - u_h\|_h$. ∎

Next we turn to investigate derivative superconvergence along special lines. More specifically, we show that, as in the linear problem [15] and also the linear mixed finite element method [16], superconvergence of $\partial u_h/\partial x$ and $\partial u_h/\partial y$ occurs along lines that pass through the Gauss points and parallel to the $y$ axis and $x$ axis, respectively.

We introduce the following seminorms for sufficiently smooth $v$:

$$\begin{aligned} [v]_1 &= \left\{\sum_e \sum_k \hat{w}_k \frac{k_e}{h_e} \int_{-1}^{1} \left(\frac{\partial \hat{v}}{\partial \xi}(\zeta_k, s)\right)^2 ds\right\}^{1/2}, \\ [v]_2 &= \left\{\sum_e \sum_k \hat{w}_k \frac{h_e}{k_e} \int_{-1}^{1} \frac{\partial \hat{v}}{\partial \eta}(s, \zeta_k)^2 ds\right\}^{1/2}, \end{aligned} \tag{38}$$

where the second sum is performed over all the Gauss points lying on a straight line parallel to one of the coordinate axes, $\hat{w}_k$ are the weights, and $\zeta_k$ are the Gauss points on a one-dimensional reference element. In our case $\hat{w}_k = 1$ and $\zeta_k = \pm(\sqrt{3}/3)$.

**Theorem 3.** Let $u \in H^4(\Omega)$, then

$$[u - u_h]_i \le Ch^3\|u\|_4, \qquad i = 1,2 \tag{39}$$

and

$$[u - Pu]_i \le Ch^3(|u|_3 + |u|_4), \qquad i = 1,2. \tag{40}$$

**Proof.** Again we estimate $[u - u_I]_1$ and $[u_I - u_h]_1$ and then apply the triangle inequality. Let $(\zeta_k, s)$, $-1 \le s \le 1$, be fixed and set $L_3(\hat{u}) = (\partial/\partial\xi)(\hat{u} - \hat{u}_I)(\zeta_k, s)$. This is clearly a linear functional of $\hat{u} \in H^4(\hat{e})$ with $|L_3(\hat{u})| \le |\hat{u} - \hat{u}_I|_{1,\infty,\hat{e}} \le C\|\hat{u}\|_{4,\hat{e}}$. If $\hat{u}$ is a quadratic polynomial, $\hat{u}_I \equiv \hat{u}$ and

$L_3(\hat{u}) \equiv 0$. If $\hat{u} = \xi^3$, then $\hat{u}_I = \xi$ and $L_3(\hat{u}) = (3\xi^2 - 1)|_{(\zeta_k,s)} = 0$, since $\zeta_k$ are roots of the Legendre polynomial $3\xi^2 - 1$. If $\hat{u} = \eta^3$, $\hat{u}_I = \eta$ and $L_3(\hat{u}) \equiv 0$. Thus $L_3(\hat{u})$ vanishes identically for cubic polynomials $\hat{u}$, and once again applying the Bramble-Hilbert lemma, we get

$$|L_3(\hat{u})| \leq C|\hat{u}|_{4,\hat{e}}. \tag{41}$$

Thus,

$$\begin{aligned}
[u - u_I]_1 &= \left\{\sum_e \sum_k \hat{w}_k \frac{k_e}{h_e} \int_{-1}^{1} [L_3(\hat{u})]^2 ds\right\}^{1/2} \\
&\leq \left\{\sum_e C \int_{-1}^{1} |\hat{u}|^2_{4,\hat{e}} ds\right\}^{1/2} \\
&\leq C\left(\sum_e h^6 |u|^2_{4,e}\right)^{1/2} \\
&\leq Ch^3|u|_4.
\end{aligned} \tag{42}$$

Observe that for $v_h \in S^h$, $[v_h]_1 = \|\partial v_h/\partial x\|_0$, since the two-point Gauss rule is exact for quadratic polynomials. As $u_I - u_h \in S_0^h$, we have $[u_I - u_h]_1 = \|(\partial/\partial x)(u_I - u_h)]\|_0 \leq \|u_I - u_h\|_1$, so applying Lemma 2 we get $[u_I - u_h]_1 \leq Ch^3\|u\|_4$ and hence $[u - u_h]_1 \leq Ch^3\|u\|_4$. In a similar fashion we may show $[u - Pu]_1 \leq Ch^3(|u|_3 + |u|_4)$.

To prove the estimate in the $[\cdot]_2$ seminorm, we fix $(s, \zeta_k)$, $-1 \leq s \leq 1$, set $L_4(\hat{u}) = (\partial/\partial\eta)(\hat{u} - \hat{u}_I)(s, \zeta_k)$ and proceed as before. ∎

**Remark.** From Theorem 3 we may deduce that at the Gauss points along the sides of the elements, the tangential derivatives are superconvergent. Note also that even though we have restricted our analysis to quadratic elements in two space dimensions, these results may readily be extended to elements of higher degree and in different dimensions (see also Zlámal [19]).

## 3. APPROXIMATE BOUNDARY FLUX CALCULATIONS

Let us now consider the problem of estimating the error of the approximate boundary flux procedure (9). Subtracting eq. (9) from eq. (8) we have for all $v_h \in S^h$

$$\langle q - q_h, v_h\rangle = a(u - u_h, v_h) - (f(u) - f(u_h), v_h). \tag{43}$$

Using the global $L^2$-error estimate $\|u - u_h\|_0 \leq Ch^3\|u\|_3$ and the Lipschitz property of $f$, we get

$$|(f(u) - f(u_h), v_h)| \leq Ch^3\|u\|_3\|v_h\|_0. \tag{44}$$

In eq. (43), we write $a(u - u_h, v_h) = a(u - u_I, v_h) + a(u_I - u_h, v_h)$ and bound $a(u - u_I, v_h)$ using Lemma 1. Then applying the Cauchy-Schwarz inequality to $a(u_I - u_h, v_h)$ and invoking Lemma 2,

$$|a(u - u_h, v_h)| \leq Ch^3\|u\|_4|v_h|_1.$$

Thus,

$$|\langle q - q_h, v_h \rangle| \leq Ch^3 \|u\|_4 \|v_h\|_1 . \tag{45}$$

Let $S_\partial^h$ denote the space of functions obtained by restricting functions in $S^h$ to the boundary $\partial\Omega$. This is precisely the function space to which the approximate flux $q_h$ belongs. As $S_\partial^h$ is a subspace of $L^2(\partial\Omega)$, for each $q \in L^2(\partial\Omega)$ we may define the projected function $Rq$ in $S_\partial^h$ by the relation

$$\langle q - Rq, v_h \rangle = 0 \quad \text{for all } v_h \text{ in } S_\partial^h . \tag{46}$$

Setting $q - q_h = q - Rq + Rq - q_h$ in eq. (45) and applying eq. (46) with $v_h = Rq - q_h$, we obtain

$$\|Rq - q_h\|_{0,\partial\Omega}^2 \leq Ch^3 \|u\|_4 \|v_h\|_1 , \tag{47}$$

where $v_h$ now represents the extension of the boundary function $Rq - q_h$ to a function that is defined over $\Omega$ and is in $S^h$.

In view of the inverse inequality [18]

$$|v_h|_1 \leq Ch^{-1/2} \|v_h\|_{0,\partial\Omega} , \tag{48}$$

we may combine eqs. (47) and (48) to obtain

$$\|Rq - q_h\|_{0,\partial\Omega} \leq Ch^{(3-1/2)} \|u\|_4 . \tag{49}$$

It is not difficult to see that for the flux function $q$,

$$\begin{aligned} \|q - Rq\|_{0,\partial\Omega} &= \inf \|v_h - q\|_{0,\partial\Omega}, \qquad v_h \in S_\partial^h \\ &\leq Ch^{5/2} \|q\|_{5/2,\partial\Omega} \\ &\leq Ch^{5/2} \|u\|_4 . \end{aligned} \tag{50}$$

Thus, we may combine these results to obtain the following:

**Theorem 4.** For $u \in H^4(\Omega)$, the error of the approximate flux computed from (9) may be estimated by

$$\|q - q_h\|_{0,\partial\Omega} \leq Ch^{5/2} \|u\|_4 . \tag{51}$$

∎

**Remarks.** This result is not entirely satisfactory since in computations, the rate of convergence of the flux approximation has been observed to be $O(h^3)$ for biquadratic elements, indicating that the above result is probably suboptimal. Note, however, that it is still a superconvergence result since the global rate of convergence of $q_h$ is expected to be $O(h^{3/2})$ only.

## 4. EVALUATION OF INTEGRALS

In many applications, integrals of the form $F(u) = \int_\Omega u\psi\, dx$ or $G(u) = \int_\Omega \nabla u \cdot \Psi\, dx$ must be evaluated where $u$ is the weak solution of eq. (4) and $\psi$ and $\Psi$ are sufficiently smooth functions. To estimate the error

$|F(u) - F(u_h)|$ it is convenient to have at our disposal negative norm estimates for $e = u - u_h$.

For an integer $s \geq 0$, we have by definition

$$\|e\|_{-s} = \sup \frac{(e,\phi)}{\|\phi\|_s}, \qquad \phi \in H^s(\Omega)\backslash\{0\}. \tag{52}$$

Thus, for $\phi \in H^s(\Omega)\backslash\{0\}$, we seek to establish a bound for $(e,\phi)$. Before doing so, let us recall that if $u$ is the weak solution of eq. (4) and $u_h$ the finite element approximation in eq. (5), then by the mean value theorem we have the following orthogonality relation:

$$\begin{aligned} a(u - u_h, v_h) &- (f(u) - f(u_h), v_h) \\ &= a(u - u_h, v_h) - \left(\left(\int_0^1 \frac{\partial f}{\partial u}[tu + (1-t)u_h]\,dt\right)(u - u_h), v_h\right) \\ &= a(e, v_h) - (h(x)e, v_h) \\ &= 0 \quad \text{for all } v_h \text{ in } S_0^h. \end{aligned} \tag{53}$$

Let $\phi \in H^s(\Omega)$, be the data in the auxiliary problem

$$L\psi - g(x)\psi = \phi \text{ in } \Omega, \tag{54}$$

$$\psi = 0 \text{ on } \partial\Omega, \tag{55}$$

with $L$ the linear elliptic operator in eq. (1) and $g(x) = (\partial f/\partial u)[u(x)]$, $u$ being the weak solution of eq. (4). Then

$$\begin{aligned} (e,\phi) &= [e, L\psi - g(x)\psi] \\ &= a(e,\psi) - (eg(x),\psi). \end{aligned} \tag{56}$$

Let $\psi_h$ be an arbitrary element in $S_0^h$. Setting $v_h = \psi_h$ in the orthogonality relation (53) and subtracting from eq. (56),

$$\begin{aligned} (e,\phi) &= a(e, \psi - \psi_h) - ([g(x) - h(x)]e, \psi) + (h(x)e, \psi_h - \psi) \\ &\leq C|e|_1|\psi - \psi_h|_1 + |([g(x) - h(x)]e, \psi)| + |(h(x)e, \psi - \psi_h)|. \end{aligned} \tag{57}$$

Now

$$\begin{aligned} |g(x) - h(x)| &= \left|\int_0^1 \left[\frac{\partial f}{\partial u}(u) - \frac{\partial f}{\partial u}[tu + (1-t)u_h]\right] dt\right| \\ &\leq \left[\int_0^1 \left|\frac{\partial^2 f}{\partial u^2}(u^*)\right|(1-t)\,dt\right]|u - u_h|. \end{aligned} \tag{58}$$

So under the additional assumption that $\partial f/\partial u$ is uniformly Lipschitz continuous, we have $|g(x) - h(x)| \leq C|u - u_h|$ and, thus, by the imbedding theorem for dimension $n \leq 3$,

$$|([g(x) - h(x)]e, \psi)| \leq C|(e^2, \psi)| \leq C\|e\|_{0,4}^2\|\psi\|_0 \leq C\|e\|_1^2\|\psi\|_0. \tag{59}$$

Also, as $|h(x)| \leq C$ for all $x$ due to the uniform Lipschitz bound on $\partial f/\partial u$,

$$|(h(x)e, \psi - \psi_h)| \leq C|e|_1|\psi - \psi_h|_1 . \tag{60}$$

Next, let $\psi_h$ be the finite element solution of the variational problem

$$a(\psi_h, v_h) - [g(x)\psi_h, v_h] = (\phi, v_h) \quad \text{for all } v_h \text{ in } S_0^h$$

associated with the linear auxiliary problem (54), (55). Assuming that $u$ is sufficiently smooth that $\psi \in H^{s+2}(\Omega) \cap H_0^1(\Omega)$, we have the error bound

$$\|\psi - \psi_h\|_1 \leq Ch^{s+1}\|\psi\|_{s+2} \leq Ch^{s+1}\|\phi\|_s . \tag{61}$$

From eqs. (57)–(61),

$$(e, \phi) \leq Ch^{s+1}|e|_1\|\phi\|_s + C\|e\|_1^2\|\phi\|_0 .$$

If $S^h$ consists of elements of degree $k$, then $s + 1 \leq k$ and

$$(e, \phi) \leq Ch^{k+s+1}\|u\|_{k+1}\|\phi\|_s + Ch^{2k}\|u\|_{k+1}^2\|\phi\|_0 .$$

Thus for $0 \leq s \leq k - 1$, we have

$$\|e\|_{-s} \leq Ch^{k+s+1}\|u\|_{k+1} + Ch^{2k}\|u\|_{k+1}^2 . \tag{62}$$

In particular, the $L^2$-norm estimate is $O(h^{k+1})$, in accordance with the corresponding estimate for linear problems. We summarize the result as follows:

**Theorem 5.** Let $u$ be the solution of eq. (4) with $u \in H^{k+1}(\Omega)$. Let $S^h$ be the space of finite elements of degree $k$. If, in addition to the properties (i)–(iv) listed for the boundary value problem (1), (2), the nonlinear forcing term $f$ is such that $\partial f/\partial u$ is uniformly Lipschitz continuous in $\mathbb{R}$, then for $0 \leq s \leq k - 1$, we have

$$\|e\|_{-s} = O(h^{k+s+1}) . \qquad \blacksquare$$

Returning to the problem of estimating the error in approximating the integral $F(u)$ by $F(u_h)$, we have:

**Corollary 1.** Let $\psi \in H^s(\Omega)$ and let $S^h$ be a finite element space of degree $k$. Let $F(u) = \int_\Omega u\psi \, dx$. For $0 \leq s \leq k - 1$, we have

$$|F(u) - F(u_h)| \leq Ch^{k+s+1}\|\psi\|_s\|u\|_{k+1} .$$

**Proof.** $|F(u) - F(u_h)| = |(u - u_h, \psi)| \leq C\|u - u_h\|_{-s}\|\psi\|_s$. Applying Theorem 5, we have the desired result. $\blacksquare$

**Remark.** The approximation $F(u_h)$ is superconvergent whenever $s \geq 1$. Note that if $\psi$ is very smooth, then $s = k - 1$ and we have the maximum rate $|F(u) - F(u_h) \leq O(h^{2k})$.

**Corollary 2.** Let $\Psi \in [H^{s+1}(\Omega)]^\nu$ and let $S^h$ be a finite element space of degree $k$. Let $G(u) = \int_\Omega \nabla u \cdot \Psi \, dx$. For $0 \le s \le k-1$,

$$|G(u) - G(u_h)| \le Ch^{k+s+1}\|\Psi\|_{s+1}\|u\|_{k+1}. \tag{63}$$

**Proof.** Applying integration by parts,

$$G(u) - G(u_h) = \int_\Omega \nabla(u - u_h) \cdot \Psi \, dx$$
$$= \int_{\partial\Omega} (u - u_h)\Psi \cdot \mathbf{n} \, ds - \int_\Omega (u - u_h)\nabla \cdot \Psi \, dx,$$

where $\mathbf{n}$ is the outward unit normal on $\partial\Omega$. Thus,

$$\begin{aligned}|G(u) - G(u_h)| &\le \|u - u_h\|_{-s-1/2,\partial\Omega}\|\Psi \cdot \mathbf{n}\|_{s+1/2,\partial\Omega} \\ &\quad + \|u - u_h\|_{-s}\|\nabla \cdot \Psi\|_s \\ &\le C\|u - u_h\|_{-s,\Omega}\|\Psi\|_{s+1}.\end{aligned} \tag{64}$$

The last inequality is obtained using the imbedding $H^{s+1}(\Omega) \hookrightarrow H^{s+1/2}(\partial\Omega)$. Now, by applying Theorem 5, we obtain the estimate (62). ∎

## 5. CONCLUSIONS

Under appropriate assumptions on the solution $u$ and nonlinear function $f(u)$ we determine error estimates involving the finite element approximation to a class of second-order elliptic semilinear problems. In particular, the respective first derivatives are shown to remain superconvergent along Gauss lines in a discretization of rectangular elements. The analysis is extended to include postprocessing formulas for the boundary flux and evaluation of integrals. Thus, one can extend the superconvergence theory to this class of semilinear problems under the stated assumptions.

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# Elliptic Systems for a Medium with Micro-Structure

R.E. SHOWALTER* and N.J. WALKINGTON

Department of Mathematics
University of Texas
Austin, TX 78712

Department of Mathematics
Carnegie-Mellon University
Pittsburgh, PA 15213

## 1. Introduction

We consider boundary-value problems for degenerate elliptic systems of the form

$$(1.1.a)\quad a(u) - \vec{\nabla}\cdot\widetilde{A}(x,\vec{\nabla}u) + \int_{\Gamma_x}\widetilde{B}(x,s,\vec{\nabla}_y U)\cdot\vec{\nu}\,ds \ni f\,, \qquad x\in\Omega\,,$$

$$(1.1.b)\quad b(U) - \vec{\nabla}_y\cdot\widetilde{B}(x,y,\vec{\nabla}_y U) \ni F\,, \qquad y\in\Omega_x\,,$$

$$(1.1.c)\quad \widetilde{B}(x,y,\vec{\nabla}_y U)\cdot\vec{\nu} + \mu(U(x,y)-u(x)) \ni 0\,, \qquad y\in\Gamma_x\,.$$

Here $\Omega$ is a domain in $\mathbb{R}^n$ and for each value of the macro-variable $x\in\Omega$ is specified a domain $\Omega_x$ with boundary $\Gamma_x$ for the micro-variable $y\in\Omega_x$. Each of $a,b,\mu$ is a maximal monotone graph. These graphs are not necessarily strictly increasing; they may be piecewise constant or multi-valued. The elliptic operators in (1.1.a) and (1.1.b) are nonlinear in the gradient of degree $p-1>0$ and $q-1>0$, respectively, with $\frac{1}{q}+\frac{1}{n}\geq\frac{1}{p}$, so some specific degeneracy is also permitted here. Certain first order spatial derivatives can be added to (1.1.a) and (1.1.b) with no difficulty, and corresponding problems with constraints, *i.e.*,

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variational inequalities, can be treated similarly. A particular example important for applications is the linear constraint

$$(1.1.c') \qquad U(x,y) = u(x)\,, \qquad y \in \Gamma_x\,,\ x \in \Omega$$

which then replaces (1.1.c). The system (1.1) with $\mu(s) = \frac{1}{\varepsilon}|s|^{q-2}s$ is called a *regularized micro-structure model*, and (1.1.a), (1.1.b), (1.1.c′) is the corresponding *matched micro-structure model* in which (formally) $\varepsilon \to 0$.

The time-dependent form of such a system arises as a model for the flow of a fluid (liquid or gas) through a fractured medium. This is assumed to be a structure of porous and permeable blocks or cells which are separated from each other by a highly developed system of fissures. The majority of fluid transport will occur along flow paths through the fissure system, and the relative volume of the cell structure is much larger than that of the fissure system. There is assumed to be no direct flow between adjacent cells, since they are individually isolated by the fissures, but the dynamics of the flux exchanged between each cell and its surrounding fissures is a major aspect of the model. The distributed micro-structure models that we develop here contain explicitly the local geometry of the cell matrix at each point of the fissure system, and they thereby reflect more accurately the flux exchange on the micro-scale of the individual cells across their intricate interface. In such a context, (1.1.a) prescribes the flow on the global scale of the fissure system and (1.1.b) gives the flow on the microscale of the individual cell at a specific point $x$ in the fissure system. The transfer of fluid between the cells and surrounding medium is prescribed by (1.1.c) or (1.1.c′). A major objective is to accurately model this fluid exchange between the cells and fissures.

The plan of this paper is as follows. In Section 2 we shall give the precise description and resolution of the stationary problem in a variational formulation by monotone operators from Banach spaces to their duals. In order to achieve this we describe first the relevant Sobolev spaces, the continuous direct sums of these spaces, and the distributed trace and constant functionals which occur in the system. The operators are monotone functions or multi-valued subgradients and serve as models for nonlinear elliptic equations in divergence form. In Section 3 we develop an abstract Green's theorem to describe the resolution of the variational form as the sum of a partial differential equation and a complementary boundary operator. Then sufficient conditions of coercivity type are given to assert the existence of generalized solutions of the variational equations.

Systems of the form (1.1) were developed in Rosen (1952), Rosen and Winshe (1950), Diesler and Wilhelm (1953) in physical chemistry as models for diffusion through a medium with a prescribed microstructure. Similar systems arose in soil science from Barker (1985), van Genuchten and Dalton (1986) and in reservoir models for fractured media in Douglas, et al. (1987), Hornung (1988). By homogenization methods such systems are obtained as limits of exact micro-scale models, and then the effective coefficients are computed explicitly from local material properties in Vogt (1982), Hornung and Jäger (to appear), Arbogast, Douglas and Hornung (1990). An existence-uniqueness theory for these linear problems which exploits the strong parabolic structure of the system was given in Showalter and Walkington (1991). One can alternatively eliminate $U$ and obtain a single functional differential equation for $u$ in the simpler space $L^2(\Omega)$, but the structure of the equation then obstructs the optimal parabolic type results; see Hornung and Showalter (1990). Also see Friedman and Tzavaras (1987) for a nonlinear system with reaction-diffusion local effects.

## 2. The Variational Formulation

We shall resolve our systems as monotone operator equations. Let $\Omega$ be a bounded domain in $\mathbb{R}^r$ with smooth boundary, $\Gamma = \partial\Omega$. Let $1 \leq p < \infty$ and denote by $L^p(\Omega)$ the space of $p^{th}$ power-integrable functions on $\Omega$, by $L^\infty(\Omega)$ the essentially bounded measurable functions, and the duality pairing by

$$(u,f)_{L(\Omega)} = \int_\Omega u(x)f(x)\,dx \ , \qquad u \in L^p(\Omega) \ , \ f \in L^{p'}(\Omega) \ ,$$

for any pair of conjugate powers, $\frac{1}{p} + \frac{1}{p'} = 1$. Let $C_0^\infty(\Omega)$ denote the space of infinitely differentiable functions with compact support in $\Omega$. $W^{m,p}(\Omega)$ is the Banach space of functions in $L^p(\Omega)$ for which each partial derivative up to order $m$ belongs to $L^p(\Omega)$, and $W_0^{m,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$. See Adams (1975) for information on these Sobolev spaces. In addition, we shall be given for each $x \in \Omega$ a bounded domain $\Omega_x$ which lies locally on one side of its smooth boundary $\Gamma_x$. Let $1 < q < \infty$ and denote by $\gamma_x : W^{1,q}(\Omega_x) \to L^q(\Gamma_x)$ the *trace* map which assigns boundary values. Let $T_x$ be the range of $\gamma_x$; this is a Banach space with the norm induced by $\gamma_x$ from $W^{1,q}(\Omega_x)$. Since $\Gamma_x$ is smooth, there is a unit outward normal $\nu_x(s)$ at each $s \in \Gamma_x$. Finally, we define $W_x^{1,q}(\Omega_x)$ to be that closed subspace consisting of those $\varphi \in W^{1,q}(\Omega_x)$ with $\gamma_x\varphi \in \mathbb{R}$, *i.e.*, each $\gamma_x(\varphi)$ is *constant* a.e. on $\Gamma_x$. We shall denote by $\vec{\nabla}_y$ the gradient on $W^{1,q}(\Omega_x)$ and by $\vec{\nabla}$ the gradient on $W^{1,p}(\Omega)$.

In order to prescribe a measurable family of cells, $\{\Omega_x, x \in \Omega\}$, set $S = \mathbb{R}^n$, let $Q \subset \Omega \times \mathbb{R}^n$ be a given measurable set, and set $\Omega_x = \{y \in \mathbb{R}^n : (x,y) \in Q\}$. Each $\Omega_x$ is measurable in $\mathbb{R}^n$ and by zero-extension we identify $L^q(Q) \hookrightarrow L^q(\Omega \times \mathbb{R}^n)$ and each $L^q(\Omega_x) \hookrightarrow L^q(\mathbb{R}^n)$. Thus we obtain

$$L^q(Q) \cong \left\{ U \in L^q(\Omega, L^q(\mathbb{R}^n)) : U(x) \in L^q(\Omega_x) \ , \ \text{a.e. } x \in \Omega \right\} .$$

We shall denote the duality on this Banach space by

$$(U,\Phi)_{L(Q)} = \int_\Omega \left\{ \int_{\Omega_x} U(x,y)\Phi(x,y)\,dy \right\} dx \ , \qquad U \in L^q(Q) \ , \ \Phi \in L^{q'}(Q) \ .$$

The *state space* for our problems will be the product $L^1(\Omega) \times L^1(Q)$.

Note that $W^{1,q}(\Omega_x)$ is continuously imbedded in $L^q(\Omega_x)$, uniformly for $x \in$

$\Omega$. It follows that the direct sum

$$\mathcal{W}_q \equiv L^q(\Omega, W^{1,q}(\Omega_x)) \equiv \{U \in L^q(Q) : U(x) \in W^{1,q}(\Omega_x) \text{ , a.e. } x \in \Omega \text{ , and } \int_\Omega \|U(x)\|^q_{W^{1,q}} dx < \infty\}$$

is a Banach space. We shall use a variety of such spaces which can be constructed in this manner. Moreover we shall assume that each $\Omega_x$ is a bounded domain in $\mathbb{R}^n$ which lies locally on one side of its boundary, $\Gamma_x$, and $\Gamma_x$ is a $C^2$-manifold of dimension $n-1$. We assume the trace maps $\gamma_x : W^{1,q}(\Omega_x) \to L^q(\Gamma_x)$ are *uniformly* bounded. Thus for each $U \in \mathcal{W}_q$ it follows that the *distributed trace* $\gamma(U)$ defined by $\gamma(U)(x,s) \equiv (\gamma_x(U(x))(s)$, $s \in \Gamma_x$, $x \in \Omega$, belongs to $L^q(\Omega, L^q(\Gamma_x))$. The distributed trace $\gamma$ maps $\mathcal{W}_q$ onto $\mathcal{T}_q \equiv L^q(\Omega, T_x) \hookrightarrow L^q(\Omega, L^q(\Gamma_x))$.

Next consider the collection $\{W_x^{1,q}(\Omega_x) : x \in \Omega\}$ of Sobolev spaces given above and denote by $\mathcal{W}_1 \equiv L^q(\Omega, W_x^{1,q}(\Omega_x))$ the corresponding direct sum. Thus for each $U \in \mathcal{W}_1$ it follows that the distributed trace $\gamma(U)$ belongs to $L^q(\Omega)$. We define $\mathcal{W}_0^{1,p}$ to be the subspace of those $U \in \mathcal{W}_1$ for which $\gamma(U) \in W_0^{1,p}(\Omega)$. Since $\gamma : \mathcal{W}_1 \to L^q(\Omega)$ is continuous, $\mathcal{W}_0^{1,p}$ is complete with the norm

$$\|U\|_{\mathcal{W}_0^{1,p}} \equiv \|U\|_{\mathcal{W}_q} + \|\gamma U\|_{W_0^{1,p}} .$$

This Banach space $W_0^{1,p}(\Omega) \times \mathcal{W}_q$ will be the *energy space* for the regularized problem (1.1) and $\mathcal{W}_0^{1,p}$ will be the energy space for the constrained problem in which (1.1.c) is replaced by the Dirichlet condition (1.1.c′). Note that $\mathcal{W}_0^{1,p}$ is identified with the closed subspace $\{[\gamma U, U] : U \in \mathcal{W}_0^{1,p}\}$ of $W_0^{1,p}(\Omega) \times \mathcal{W}_q$. Finally, we shall let $\mathcal{W}_0$ denote the kernel of $\gamma$, $\mathcal{W}_0 = \{U \in \mathcal{W}_q : \gamma U = 0 \text{ in } \mathcal{T}_q\}$.

We have defined $W_x^{1,q}(\Omega_x)$ to be the set of $w \in W^{1,q}(\Omega_x)$ for which $\gamma_x w$ is a constant multiple of $\mathbf{1}_x$, the constant function equal to one on $\Gamma_x$. Thus $W_x^{1,q}(\Omega_x)$ is the pre-image by $\gamma_x$ of the subspace $\mathbb{R} \cdot \mathbf{1}_x$ of $T_x$. We specified the subspace $\mathcal{W}_1$ similarly as the subspace of $\mathcal{W}_q$ obtained as the pre-image by $\gamma$ of the subspace $L^q(\Omega)$ of $\mathcal{T}_q$. To be precise, we denote by $\lambda$ the map of $L^q(\Omega)$ into

$\mathcal{T}_q$ given by $\lambda v(x) = v(x) \cdot \mathbf{1}_x$, a.e. $x \in \Omega$, $v \in L^q(\Omega)$; $\lambda$ is an isomorphism of $L^q(\Omega)$ onto a closed subspace of $\mathcal{T}_q$. The dual map $\lambda'$ taking $\mathcal{T}_q'$ into $L^{q'}(\Omega)$ is given by

$$\lambda' g(v) = g(\lambda v) = \int_\Omega g_x(\mathbf{1}_x) \cdot v(x)\,dx\ , \qquad g \in \mathcal{T}_q'\ ,\ v \in L^q(\Omega)\ ,$$

so we have $\lambda' g(x) = g_x(\mathbf{1}_x)$, a.e. $x \in \Omega$. Moreover, when $g_x \in L^{q'}(\Gamma_x)$ it follows that

$$g_x(\mathbf{1}_x) = \int_{\Gamma_x} g_x(y)\,dy\ ,$$

the integral of the indicated boundary functional. Thus, for $g \in L^{q'}(\Omega, L^{q'}(\Gamma_x)) \subset \mathcal{T}_q'$, $\lambda' g \in L^{q'}(\Omega)$ is given by

$$\lambda' g(x) = \int_{\Gamma_x} g_x(y)\,dy\ , \quad \text{a.e. } x \in \Omega\ . \tag{2.1}$$

The imbedding $\lambda$ of $L^q(\Omega)$ into $\mathcal{T}_q$ and its dual map $\lambda$ will play an essential role in our system below.

We construct the elliptic differential operators in divergence form as realizations of monotone operators from Banach spaces to their duals. Assume we are given $\widetilde{A} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ such that for some $1 < p < \infty$, $g_1 \in L^{p'}(\Omega)$, $g_0 \in L^1(\Omega)$, $c$ and $c_0 > 0$

(2.2.a) $\widetilde{A}(x, \vec{\xi})$ is continuous in $\vec{\xi} \in \mathbb{R}^n$ and measurable in $x$, and

$$|\widetilde{A}(x, \vec{\xi})| \le c|\xi|^{p-1} + g_1(x)\ ,$$

(2.2.b) $\langle \widetilde{A}(x, \vec{\xi}) - \widetilde{A}(x, \vec{\eta}), \vec{\xi} - \vec{\eta} \rangle \ge 0\ ,$

(2.2.c) $\widetilde{A}(x, \vec{\xi}) \cdot \vec{\xi} \ge c_0 |\vec{\xi}|^p - g_0(x)$

for a.e. $x \in \Omega$ and all $\vec{\xi}, \vec{\eta} \in \mathbb{R}^n$.

Then the global diffusion operator $\mathcal{A} : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is given by

$$\mathcal{A}u(v) = \int_\Omega \widetilde{A}(x, \vec{\nabla} u(x)) \vec{\nabla} v(x)\,dx\ , \qquad u, v \in W_0^{1,p}(\Omega)\ .$$

Thus, each $\mathcal{A}u$ is equivalent to its restriction to $C_0^\infty(\Omega)$, the distribution

$$Au \equiv \mathcal{A}u\big|_{C_0^\infty(\Omega)} = -\vec{\nabla}\cdot\widetilde{A}(\cdot,\vec{\nabla}u)$$

which specifies the value of this nonlinear elliptic divergence operator.

Assume we are given $\widetilde{B} : Q \times \mathbb{R}^n \to \mathbb{R}^n$ such that for some $1 < q < \infty$, $h_1 \in L^{q'}(Q)$, $h_0 \in L^1(Q)$, $c$ and $c_0 > 0$

(2.3.a) $\widetilde{B}(x,y,\vec{\xi})$ is continuous in $\vec{\xi} \in \mathbb{R}^n$ and measurable in $(x,y) \in Q$, and

$$|\widetilde{B}(x,y,\vec{\xi})| \le c|\vec{\xi}|^{q-1} + h_1(x,y)\ ,$$

(2.3.b) $\langle \widetilde{B}(x,y,\vec{\xi}) - \widetilde{B}(x,y,\vec{\eta}),\ \vec{\xi} - \vec{\eta}\rangle \ge 0\ ,$

(2.3.c) $\widetilde{B}(x,y,\vec{\xi})\cdot\vec{\xi} \ge c_0|\vec{\xi}|^q - h_0(x,y)$

for a.e. $(x,y) \in Q$ and all $\vec{\xi},\vec{\eta} \in \mathbb{R}^n$.

Then define for each $x \in \Omega$, $\mathcal{B}_x : W^{1,q}(\Omega_x) \to W^{1,q}(\Omega_x)'$ by

$$\mathcal{B}_x w(v) = \int_{\Omega_x} \widetilde{B}(x,y,\vec{\nabla}_y w(y))\vec{\nabla}_y v(y)\,dy\ , \qquad w,v \in W^{1,q}(\Omega_x)\ .$$

The elliptic differential operator on $\Omega_x$ is given by the *formal part* of $\mathcal{B}_x$, the distribution

$$B_x w \equiv \mathcal{B}_x w\big|_{C_0^\infty(\Omega_x)} = -\vec{\nabla}_y\cdot\widetilde{B}(x,\cdot,\vec{\nabla}_y w)$$

in $W_0^{1,q}(\Omega_x)'$. Also, we shall denote by $\mathcal{B} : \mathcal{W}_q \to \mathcal{W}_q'$ the ***distributed*** operator constructed from the collection $\{\mathcal{B}_x : x \in \Omega\}$ by

$$\mathcal{B}U(x) = \mathcal{B}_x(U(x))\ , \qquad \text{a.e. } x \in \Omega\ ,\ U \in \mathcal{W}_q\ ,$$

and we note that this is equivalent to

$$\mathcal{B}U(V) \equiv \int_\Omega \mathcal{B}_x(U(x))V(x)\,dx\ , \qquad U,V \in \mathcal{W}_q\ .$$

The exchange term in our system will be given as a monotone graph which is a subgradient operator. Thus, assume $m : \mathbb{R} \to \mathbb{R}^+$ is convex and bounded by

$$m(s) \le C(|s|^q + 1)\ , \qquad s \in \mathbb{R}\ , \tag{2.4}$$

hence, continuous. Then by

$$\tilde{m}(g) \equiv \int_\Omega \int_{\Gamma_x} m(g(x,s))\, ds\, dx\ , \qquad g \in L^q(\Omega, L^q(\Gamma_x))$$

we obtain the convex, continuous $\tilde{m} : L^q(\Omega, L^q(\Gamma_x)) \to \mathbb{R}^+$. Assume $\frac{1}{q} + \frac{1}{n} \geq \frac{1}{p}$ so that $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, and consider the linear continuous maps

$$\lambda : W_0^{1,p}(\Omega) \to L^q(\Omega, L^q(\Gamma_x))\ , \quad \gamma : \mathcal{W}_q \to L^q(\Omega, L^q(\Gamma_x))\ .$$

Then the composite function

$$M[u, U] \equiv \tilde{m}(\gamma U - \lambda u)\ , \qquad u \in W_0^{1,p}(\Omega)\ ,\ U \in \mathcal{W}_q\ ,$$

is convex and continuous on $W_0^{1,p}(\Omega) \times \mathcal{W}_q$. The subgradients are directly computed by standard results Ekland and Temam (1976). Specifically, we have $\hat{g} \in \partial\tilde{m}(g)$ if and only if

$$\hat{g}(x,s) \in \partial m(g(x,s))\ , \quad \text{a.e. } s \in \Gamma_x\ , \ \text{a.e. } x \in \Omega\ ,$$

and we have $[f, F] \in \partial M[u, U]$ if and only if $f = -\lambda'(\mu)$ in $W^{-1,p'}(\Omega)$ and $F = \gamma'(\mu)$ in $\mathcal{W}_q'$ for some $\mu \in \partial\tilde{m}(\gamma U - \lambda u)$.

The following result gives sufficient conditions for the *regularized problem* to be well-posed.

**Theorem 1.** *Assume $1 < p, q$, $\frac{1}{q} + \frac{1}{n} \geq \frac{1}{p}$, and define the spaces and operators $\lambda, \gamma$ as above. Specifically, the sets $\{\Omega_x : x \in \Omega\}$ are uniformly bounded with smooth boundaries, and the trace maps $\{\gamma_x\}$ are uniformly bounded. Let the functions $\widetilde{A}, \widetilde{B}$, and $m$ satisfy (2.2), (2.3) (2.4), and assume in addition that*

$$m(s) \geq c_0 |s|^q\ , \qquad s \in \mathbb{R}\ . \tag{2.5}$$

*Then for each pair $f \in W^{-1,p'}(\Omega)$, $F \in \mathcal{W}_q'$ there exists a solution of*

(2.6.a) $\qquad u \in W_0^{1,p}(\Omega) : A(u) - \lambda'(\mu) = f \ \text{ in } \ W^{-1,p'}(\Omega)$

(2.6.b) $\qquad U \in \mathcal{W}_q : B(U) + \gamma'(\mu) = F \ \text{ in } \mathcal{W}_q'$

(2.6.c) $\qquad \mu \in L^{q'}(\Omega, L^{q'}(\Gamma_x)) : \mu \in \partial\tilde{m}(\gamma U - \lambda u)\ .$

*For any such solution we have*

$$\int_{\Gamma_x} \mu(x,s)\,ds = \langle F(x), \mathbf{1}_x \rangle\,, \qquad \text{a.e. } x \in \Omega\,, \tag{2.7}$$

*where* $\mathbf{1}_x$ *denotes the constant unit function in* $W^{1,q}(\Omega_x)$.

*Proof.* The system (2.6) is a "pseudo-monotone plus subgradient" operator equation of the form

$$\begin{aligned} &[u,U] \in W_0^{1,p}(\Omega) \times \mathcal{W}_q : \quad \text{for all } [v,V] \in W_0^{1,p}(\Omega) \times \mathcal{W}_q \\ &\mathcal{A}u(v) + \mathcal{B}U(V) + \partial\mathcal{M}[u,U]([v,V]) \ni f(v) + F(V)\,. \end{aligned} \tag{2.6'}$$

It remains only to verify a coercivity condition, namely,

$$\frac{\mathcal{A}u(u) + \mathcal{B}U(U) + \tilde{m}(\gamma U - \lambda u)}{\|u\|_{W_0^{1,p}(\Omega)} + \|U\|_{\mathcal{W}_q}} \to +\infty \tag{2.8}$$

$$\text{as } \|u\|_{W_0^{1,p}(\Omega)} + \|U\|_{\mathcal{W}_q} \to +\infty\,.$$

Choose $k = \max\{|y_n| : y \in \Omega_x,\ x \in \Omega\}$ and let $\nu_x = (\nu_x^1, \dots, \nu_x^n)$ be the unit normal on $\Gamma_x$. For $v \in W^{1,q}(\Omega_x)$ we have by Gauss' Theorem

$$\begin{aligned} \int_{\Omega_x} (|v|^q + y_n q |v|^{q-1}\partial_n v) &= \int_{\Omega_x} \partial_n (y_n |v(y)|^q)\,dy \\ &= \int_{\Gamma_x} \nu_x^n(s) s_n |\gamma_x v(s)|^q\,ds\,. \end{aligned}$$

Hölder's inequality then shows

$$\|v\|^q_{L^q(\Omega_x)} \le k\|\gamma_x v\|^q_{L^q(\Gamma_x)} + qk\|v\|^{q-1}_{L^q(\Omega_x)}\|\partial_n v\|_{L^q(\Omega_x)}\,,$$

and from this follows

$$\|v\|^q_{L^q(\Omega_x)} \le 2k\|\gamma_x v\|^q_{L^q(\Gamma_x)} + (2k)^q (q-1)^{q-1}\|\partial_n v\|^q_{L^q(\Omega_x)}$$

by Young's inequality. From here we obtain

$$c_0\|V\|^q_{L^q(Q)} \le \|\gamma V\|^q_{L^q(\Omega, L^q(\Gamma_x))} + \|\nabla_y V\|^q_{L^q(Q)}\,, \quad V \in \mathcal{W}_q\,. \tag{2.9}$$

Thus from the a-priori estimate

$$\mathcal{A}u(u) + \mathcal{B}U(U) + \mathcal{M}(\gamma U - \lambda u) \geq$$

$$c_0\|\vec{\nabla} u\|^p_{L^p(\Omega)} - \|g_0\|_{L^1(\Omega)} + c_0\|\vec{\nabla}_y U\|^q_{L^q(Q)} - \|h_0\|_{L^1(Q)} + c_0\|\gamma U - \lambda u\|^q_{L^q(\Omega, L^q(\Gamma_x))}\ , \quad u \in W_0^{1,p}(\Omega)\ ,\ U \in \mathcal{W}_q\ , \tag{2.10}$$

the Poincaré-type inequality (2.9) and the equivalence of $\|\nabla u\|_{L^p(\Omega)}$ with the norm on $W_0^{1,p}(\Omega)$, we can obtain the coercivity condition (2.8). Specifically, if (2.8) is bounded by $K$, then (2.10) is bounded above by

$$K\big(\|u\|_{W_0^{1,p}(\Omega)} + \|\nabla_y U\|_{L^q(Q)} + \|\gamma U\|_{L^q(\Omega, L^q(\Gamma_x))}\big)$$

$$\leq K\big(\|u\|_{W_0^{1,p}(\Omega)} + \|\nabla_y U\|_{L^q(Q)} + \|\gamma U - \lambda\varphi\|_{L^q(\Omega, L^q(\Gamma_x))} + \|\lambda u\|_{L^q(\Omega)}\big)\ ,$$

and the last term is dominated by the first. This gives an explicit bound on each of these terms and, hence, on $\|u\|_{W_0^{1,p}(\Omega)} + \|U\|_{\mathcal{W}_q}$.

Finally, we apply (2.6.b) to the function $V \in \mathcal{W}_q$ given by $V(x, y) = v(x)$ for some $v \in L^q(\Omega)$, and this shows

$$\mu(\gamma v) = \langle F, v\rangle$$

since $\mathcal{B}U(V) = 0$, and thus

$$\int_\Omega \lambda' \mu(x) v(x)\, dx = \mu(\lambda v) = \mu(\gamma v) = \int_\Omega \langle F(x), 1\rangle v(x)\, dx\ .$$

The identity (2.7) now follows from (2.1).

The more general case (1.1) of a monotone pointwise perturbation is easily handled likewise.

**Corollary 1.** *Let $\varphi : \mathbb{R} \to \mathbb{R}^+$ and $\Phi : \mathbb{R} \to \mathbb{R}^+$ be convex and continuous, with $\varphi(0) = \Phi(0) = 0$, and assume*

$$\varphi(s) \leq C\big(|s|^q + 1\big)\ ,\quad \Phi(s) \leq C\big(|s|^q + 1\big)\ ,\qquad s \in \mathbb{R}\ . \tag{2.11}$$

*For each pair $f \in W^{-1,p'}(\Omega)$, $F \in \mathcal{W}_q'$, there exists a solution of*

$$\text{(2.12.a)} \qquad u \in W_0^{1,p}(\Omega) : a + A(u) - \lambda'(\mu) = f \ \text{ in } \ W^{-1,p'}(\Omega)$$

$$\text{(2.12.b)} \qquad U \in \mathcal{W}_q : b + \mathcal{B}(U) + \gamma'(\mu) = F \ \text{ in } \ \mathcal{W}_q'$$

$$\text{(2.12.c)} \qquad \mu \in \partial\tilde{m}(\gamma U - \lambda u) \ \text{ in } \ L^{q'}(\Omega, L^{q'}(\Gamma_x)) \,, \ \text{ and}$$

$$\text{(2.12.d)} \qquad a \in \partial\varphi(u) \ \text{ in } \ L^{q'}(\Omega) \,; \ b \in \partial\Phi(U) \ \text{ in } \ L^{q'}(Q) \,.$$

*For any such solution we have*

$$\text{(2.13)} \qquad \int_{\Omega_x} b(x,y)\,dy + \int_{\Gamma_x} \mu(x,s)\,ds = \langle F(x), 1_x \rangle \,; \quad \text{a.e. } x \in \Omega \,.$$

*Proof.* This follows as above but with the continuous convex function

$$\Psi[u,U] = \int_\Omega \varphi(u(x))\,dx + \int_\Omega \int_{\Omega_x} \Phi(U(x,y))\,dy\,dx$$
$$+ \tilde{m}(\gamma U - \lambda u) \,, \quad [u,U] \in W_0^{1,p}(\Omega) \times \mathcal{W}_q \,.$$

The subgradient can be computed termwise because the three terms are continuous on $L^q(\Omega)$, $L^q(Q)$, and $L^q(\Omega, L^q(\Gamma_x))$, respectively.

*Remark.* The lower bound (2.5) on $m(\cdot)$ may be deleted in Corollary 1 if such a lower estimate is known to hold for $\Phi$. It is also unnecessary in the matched microstructure model; see below.

## 3. The Green's Formula

In order to prescribe the boundary condition (1.1.c) explicitly, we develop an appropriate Green's formula for the operators $\mathcal{B}_x$. Note that we can identify $L^{q'}(\Omega_x) \subset W^{-1,q'}(\Omega_x)$ since $W_0^{1,q}(\Omega_x)$ is dense in $L^q(\Omega_x)$, so it is meaningful to define

$$D_x \equiv \{ w \in W^{1,q}(\Omega_x) : \mathcal{B}_x w \in L^{q'}(\Omega_x) \} \,.$$

This is the domain for the *abstract Green's Theorem.*

**Lemma 1.** *There is a unique operator $\partial_x : D_x \to T'_x$ for which $\mathcal{B}_x w = B_x w + \gamma'_x \partial_x w$ for all $w \in D_x$. That is, we have*

$$\mathcal{B}_x w(v) = (B_x w, v)_{L(\Omega_x)} + \langle \partial_x w, \gamma_x v \rangle , \qquad v \in W^{1,q}(\Omega_x) , \tag{3.1}$$

*for every $w \in D_x$.*

*Proof.* The strict morphism $\gamma_x$ of $W^{1,q}(\Omega_x)$ onto $T_x$ has a dual $\gamma'_x$ which is an isomorphism of $T'_x$ onto $W_0^{1,q}(\Omega_x)^{\perp}$, the annihilator in $W^{1,q}(\Omega_x)'$ of the kernel of $\gamma_x$. For each $w \in D_x$, the difference $\mathcal{B}_x w - B_x w$ is in $W_0^{1,q}(\Omega_x)^{\perp}$, so it is equal to $\gamma'_x(\partial_x w)$ for a unique element $\partial_x w \in T'_x$.

*Remark.* The identity (3.1) is a generalized decomposition of $\mathcal{B}_x$ into a partial differential operator on $\Omega_x$ and a boundary condition on $\Gamma_x$. If $\Gamma_x$ is smooth, $\nu_x$ denotes the unit outward normal on $\Gamma_x$, and if $\tilde{B}(x, \cdot, \vec{\nabla}_y w) \in [W^{1,q'}(\Omega_x)]^n$, then $w \in D_x$ and from the classical Green's Theorem we obtain

$$\mathcal{B}_x w(v) - (B_x w, v)_{L(\Omega_x)} = \int_{\Gamma_x} \tilde{B}(x, s, \vec{\nabla}_y w) \vec{\nu}_x(s) \gamma v(s)\, ds , \qquad v \in W^{1,q}(\Omega_x) .$$

Thus, $\partial_x w = \tilde{B}(x, \cdot, \vec{\nabla}_y w) \cdot \vec{\nu}_x$ is the indicated normal derivative in $L^{q'}(\Gamma_x)$ when $\tilde{B}(x, \cdot, \vec{\nabla}_y w)$ is as smooth as above, and so we can regard $\partial_x w$ in general as an extension of this nonlinear differential operator on the boundary.

The *formal part* of $\mathcal{B} : \mathcal{W}_q \to \mathcal{W}'_q$ is the operator $B : \mathcal{W}_q \to \mathcal{W}'_0$ given by the restriction $B(U) \equiv \mathcal{B}U|_{\mathcal{W}_0}$. Since $\mathcal{W}_0$ is dense in $L^q(Q)$ we can specify the domain

$$D \equiv \{U \in \mathcal{W}_q : B(U) \in L^{q'}(Q)\}$$

on which we obtain as before a distributed form of Green's theorem.

**Lemma 2.** *There is a unique operator $\partial : D \to T'_q$ such that*

$$\mathcal{B}(U)(V) = (B(U), V)_{L(Q)} + \langle \partial U, \gamma V \rangle , \qquad U \in D , \ V \in \mathcal{W}_q .$$

**Theorem 2.** *Let the Sobolev spaces and trace operators be given as above. We summarize them in the following diagrams*

$$\begin{array}{ccccccc}
L^q(\Omega_x) & & L^q(\Gamma_x) & \qquad & L^q(Q) & & L^q(\Omega, L^q(\Gamma_x)) \\
\cup & & \cup & & \cup & & \cup \\
W^{1,q}(\Omega_x) & \xrightarrow{\gamma_x} & T_x & & \mathcal{W}_q & \xrightarrow{\gamma} & \mathcal{T}_q \\
\cup & & \lambda_x \uparrow & & \cup & & \lambda \uparrow \\
W^{1,q}_x(\Omega_x) & \longrightarrow & \mathbf{R}\cdot\mathbf{1}_x & & \mathcal{W}_1 & \xrightarrow{\gamma_1} & L^q(\Omega) \\
\cup & & \cup & & \cup & & \uparrow \\
W^{1,q}_0(\Omega_x) & \longrightarrow & \{0\} & & \mathcal{W}_0 & \longrightarrow & \{0\}
\end{array}$$

*in which $\gamma_1$ is the restriction of $\gamma$ to $\mathcal{W}_1$. $W^{1,q}_0(\Omega_x)$, $\mathcal{W}_0$ are dense in $L^q(\Omega_x)$, $L^q(Q)$, respectively. Let operator $\mathcal{B}_x$, $x \in \Omega$, and $\mathcal{B}$ be given and define their formal parts $B_x$, $B$ as above. Then construct the domains $D_x$, $D$ and boundary operators $\partial_x$, $\partial$ as in Lemma 1 and Lemma 2, respectively. It follows that for any $U \in \mathcal{W}_q$,*

*(a) $BU(x) = \mathcal{B}_x(U(x))$ in $W^{1,q}_0(\Omega_x)'$ for a.e. $x \in \Omega$, and $U \in D$ if and only if $U(x) \in D_x$ for a.e. $x \in \Omega$ and $x \mapsto B_xU(x)$ belongs to $L^{q'}(Q)$;*

*(b) for each $U \in D$,*

$$\partial U(x) = \partial_x(U(x)) \text{ in } T_x' \text{ for a.e. } x \in \Omega$$

*and*

$$\mathcal{B}U = BU + \gamma_1'(\lambda'\partial U) \text{ in } \mathcal{W}_1' ,$$

*and for each $V \in \mathcal{W}_1$ we have*

$$\int_\Omega \mathcal{B}_xU(x)(V(x))\,dx = \int_Q B_xU(x)V(x)\,dy\,dx + \int_\Omega \langle \partial_xU(x), \mathbf{1}_x\rangle(\gamma_1V)(x)\,dx .$$

*Proof.* (a) For $V \in \mathcal{W}_0$ we obtain from the definitions of $B$, $\mathcal{B}$ and $\mathcal{B}_x$, respectively,

$$\int_\Omega BU(x)V(x)\,dx = \int_\Omega \mathcal{B}U(V)\,dx = \int_\Omega \mathcal{B}_xU(x)(V(x))\,dx = \int_\Omega B_xU(x)V(x)\,dx ,$$

and so the first equality holds since $\mathcal{W}_0' = L^{q'}(\Omega, W_0^{1,q}(\Omega_x)')$. The characterization of $D$ is immediate now.

(b) For $V \in \mathcal{W}_q$ we obtain from the definitions of $\gamma, \partial, \partial_x$, respectively, and (a)

$$\int_\Omega \partial U(\gamma_x V(x))\,dx = \int_\Omega \partial U(\gamma V)\,dx = \int_\Omega (\mathcal{B}U - BU)(x)V(x)\,dx$$

$$= \int_\Omega \partial_x(U(x))\gamma_x V(x)\,dx\ .$$

Since the range of $\gamma$ is $\mathcal{T}_q' = L^{q'}(\Omega, T_x')$, the first equality follows. The second is immediate from Lemma 2 since on $\mathcal{W}_1$, $\gamma = \lambda \circ \gamma_1$ and $\gamma' = \gamma_1'\lambda'$, and the third follows from the preceding remarks.

**Corollary 2.** *In the situation of Corollary 1, $f \in L^{q'}(\Omega)$ and $F \in L^{q'}(Q)$ if and only if $Au \in L^{q'}(\Omega)$ and $B(U) \in L^{q'}(Q)$, and in that case the solution satisfies almost everywhere*

$$a(x) \in \partial\varphi(u(x))\ ,\ a(x) + Au(x) + \int_{\Omega_x} b(x,y)\,dy = f(x) + \int_{\Omega_x} F(x,y)\,dy\ ,\quad x \in \Omega\ ,$$

$$u(s) = 0\ ,\qquad s \in \Gamma\ ,$$

$$b(x,y) \in \partial\Phi(U(x,y))\ ,\quad b(x,y) + BU(x,y) = F(x,y)\ ,\qquad y \in \Omega_x\ ,$$

$$\mu(x,s) \in \partial m(\gamma U(x,s) - u(x))\ ,\quad \partial_x(U(x))(s) + \mu(x,s) = 0\ ,\qquad s \in \Gamma_x\ .$$

Finally, we note that corresponding results for the *matched microstructure model* are obtained directly by specializing the system (2.6′) to the space $\mathcal{W}_0^{1,p}$. This is identified with $\{[\gamma U, U] : U \in \mathcal{W}_0^{1,p}\}$ as a subspace of $W_0^{1,p}(\Omega) \times \mathcal{W}_q$, and we need only to restrict the solution $[u, U]$ and the test functions $[v, V]$, $v = \gamma V$, to this subspace to resolve the matched model. Then the exchange term $\mathcal{M}$ does not occur in the system; see the proof of Proposition 1, especially for the coercivity. These observations yield the following analogous results for the matched microstructure model.

**Theorem 1′.** *Assume* $1 < p, q, \frac{1}{q} + \frac{1}{n} \geq \frac{1}{p}$, *and define the spaces and operators* $\lambda, \gamma$ *as before. Let the functions* $\widetilde{A}, \widetilde{B}$, *and* $m$ *satisfy* (2.2), (2.3) *and* (2.4). *Then for each pair* $f \in W^{-1,p'}(\Omega)$, $F \in \mathcal{W}_1'$ *there exists a unique solution of*

$$u \in W_0^{1,p}(\Omega) : A(u) = f + \langle F, 1 \rangle \quad \text{in} \quad W^{-1,p'}(\Omega) \tag{3.2.a}$$

$$U \in \mathcal{W}_1 : B(U) = F \quad \text{in} \quad \mathcal{W}_0' \tag{3.2.b}$$

$$\gamma U = \lambda u \quad \text{in} \quad L^q(\Omega) \subset \mathcal{T}_q \ . \tag{3.2.c}$$

**Corollary 1′.** *Suppose* $\varphi, \Phi$ *are given as before and assume* (2.11). *For* $f, F$ *as above there exists a unique solution of*

$$u \in W_0^{1,p}(\Omega) : a + \langle b, 1 \rangle + A(u) = f + \langle F, 1 \rangle \quad \text{in} \quad W^{-1,p'}(\Omega) \tag{3.3.a}$$

$$U \in \mathcal{W}_1 : b + B(U) = F \quad \text{in} \quad \mathcal{W}_0' \tag{3.3.b}$$

$$\gamma U = \lambda u \quad \text{in} \quad L^q(\Omega) \subset \mathcal{T}_q \tag{3.3.c}$$

$$a \in \partial\varphi(u) \quad \text{in} \quad L^{q'}(\Omega)\ , \quad b \in \partial\Phi(U) \quad \text{in} \quad L^{q'}(Q)\ . \tag{3.3.d}$$

*In addition,* $f \in L^{q'}(\Omega)$ *and* $F \in L^{q'}(Q)$ *if and only if* $Au \in L^{q'}(\Omega)$ *and* $B(U) \in L^{q'}(Q)$, *and in that case the solution satisfies almost everywhere*

$$a(x) \in \partial\varphi(u(x))\ , \quad a(x) + Au(x) + \int_{\Omega_x} b(x,y)\,dy = f(x) + \int_{\Omega_x} F(x,y)\,dy\ , \quad x \in \Omega\ ,$$

$$u(s) = 0 \qquad s \in \Gamma\ ,$$

$$b(x,y) \in \partial\Phi(U(x,y))\ , \quad b(x,y) + BU(x,y) = F(x,y)\ , \qquad y \in \Omega_x\ ,$$

$$U(x,s) = u(x)\ , \qquad s \in \Gamma_x\ .$$

*Remarks.* For the very special case of $p = q \geq 2$ and $a(u) = u$, $b(U) = U$ in the situation of Theorem 1 it follows from Brezis (1972) or Lions (1969) that the Cauchy-Dirichlet problem for (1.1) is well-posed in the space $L^p(0,T; W_0^{1,p}(\Omega) \times$

$\mathcal{W}_p$) with appropriate initial data $u(x,0)$, $U(x,y,0)$ and source functions $f(x,t)$, $F(x,y,t)$. A similar remark holds in the case of Theorem 1′ for the matched model with (1.1.c′). These restrictive assumptions will be substantially relaxed in Showalter and Walkington (to appear).

Furthermore, variational inequalities may be resolved for problems corresponding to either the regularized or the matched microstructure model by adding the indicator function of a convex constraint set to the convex function $\Psi$. Thus one can handle such problems with constraints on the global variable $u$, the local variables $U$, or their difference $\lambda u - \gamma U$ on the interface.

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# A Porous Media System with Microstructure


PETER KNABNER

Institut für Mathematik
Universität Augsburg
Universitäts-Str. 8
D8900 Augsburg, West Germany

RALPH SHOWALTER *

Department of Mathematics
The University of Texas at Austin
Austin, TX 78712
USA



**Abstract.** A model of nonlinear diffusion through a porous medium is considered, where the solute is adsorbed through the boundaries of the individual cells in the prescribed microstructure, and the flow within each cell is governed by a corresponding porous medium equation. The resulting system is shown to be well-posed in an appropriate $L^1$ space, and certain properties of the solution are obtained for special cases.


## 1. Introduction.

We begin with a description of the porous media system to be studied. Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ which lies locally on one side of its $C^2$ boundary $\Gamma$. Denote by $\Delta$ the Laplace operator in $\mathbb{R}^N$ and by $\vec{n}$ the unit outward normal on $\Gamma$. Suppose that $Q$ is a given measurable subset of $\Omega \times \mathbb{R}^N$ and set $\Omega_x \equiv \{y \in \mathbb{R}^N : [x, y] \in Q\}$. For each point $x \in \Omega$ we assume $\Omega_x$ is a domain in $\mathbb{R}^N$ with boundary $\Gamma_x$, and these satisfy (uniformly in $x$) the same hypotheses as $\Omega$ and $\Gamma$; denote by $\Delta_y$ the Laplace operator in the micro-variable $y \in \Omega_x$ and by $\vec{n}_x$ the unit outward normal on $\Gamma_x$. Suppose we are given five maximal monotone graphs in $\mathbb{R}$ denoted by $a, b, \alpha, \beta, \gamma$. We shall consider the singular-degenerate parabolic system

$$(1.1.a) \qquad \frac{\partial}{\partial t} a(w) - \Delta(w) + \int_{\Gamma_x} \vec{\nabla}_y W \cdot \vec{n}_x \, ds \ni f\,, \qquad x \in \Omega\,,$$

$$(1.1.b) \qquad \vec{\nabla} w \cdot n + b(w) \ni 0\,, \qquad x \in \Gamma\,,$$ w

$$(1.1.\alpha) \qquad \frac{\partial}{\partial t} \alpha(W) - \Delta_y W \ni F\,, \qquad y \in \Omega_x\,,$$



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$$(1.1.\beta) \qquad \vec{\nabla}_y W \cdot \vec{n}_x + \beta(W) - \gamma\big(a(w)\big) \ni \overset{0}{\cancel{\emptyset}} , \qquad s \in \Gamma_x$$

for $t > 0$ with initial conditions $a(w(x,0))$, $b(W(x,y,0))$ prescribed in addition to the sources $f \in L^1(\Omega\times(0,\infty))$ and $F \in L^1(Q\times(0,\infty))$, where $Q = \prod_{x\in\Omega}\Omega_x$. Thus (1.1.a) governs the flow on the macro-scale of the porous medium $\Omega$, *i.e.*, through the fissures in this global domain, and (1.1.$\alpha$) prescribes the flow on the micro-scale of the individual cell $\Omega_x$ located at the point $x \in \Omega$. The integral term in (1.1.a) is the total flux flowing across $\Gamma_x$ into the cell, and this flux is determined by (1.1.$\beta$).

The monotone graphs are possibly multi-valued functions, so we obtain inclusions instead of equations: the corresponding equation holds for *some* selection out of the graph. In general, $w(x)$ is the density of mobil solute at $x \in \Omega$, $W(x,y)$ is the density of solute at $y \in \Omega_x$ adsorbed in this cell, $a(w)$ and $\alpha(W)$ represent corresponding concentrations of the solute present, $b(w)$ is the flux across $\Gamma$ at a given density $w$, and $\beta(W) - \gamma(\cancel{a})$ determines the transport of flux across $\Gamma_x$ for given $\gamma(a(w))$
density $W$ on the inside of $\Omega_x$ and concentration $a(w)$ on the outside of this cell at $x$. These last two graphs correspond to the ***adsorption*** isotherms of the media.

The problem (1.1) will be regarded as an abstract Cauchy problem for the evolution equation

$$(1.2) \qquad u'(t) + A\big(u(t)\big) \ni f(t) , \qquad t > 0 ,$$

in the Banach space $X = L^1(\Omega)\times L^1(Q)$. Recall from [3], [2] that an ***integral solution*** of (1.2) in a Banach space $X$ is a $u \in C([0,\infty), X)$ such that $u(t) \in \overline{\mathrm{dom}(A)}$ and

$$\tfrac{1}{2}\|u(t) - x\| \le \tfrac{1}{2}\|u(s) - x\| + \int_s^t \big\langle f(\tau) - y, u(\tau) - x\big\rangle \, d\tau$$

for each $y \in A(x)$ and $0 \le s \le t$. The pairing in the integral is the semi-scalar-product

$$\langle y, x\rangle = \sup\big\{(y, x^*) : x^* \in X^* \ , \ \|x^*\| = \|x\| = x^*(x)\big\} \ .$$

A (possibly multi-valued) operator $A$ on $X$ is called *accretive* if

$$\|u_1 - u_2\| \le \|u_1 - u_2 + \varepsilon(f_1 - f_2)\|$$

for all $\varepsilon > 0$, $[u_1, f_1] \in A$, $[u_2, f_2] \in A$. If also $I + A$ is onto $X$ then $A$ is called *m-accretive*. The fundamental result of [3] is that for $f \in L^1([0, \infty), X)$ and $u_0 \in \overline{\text{dom}(A)}$ there is a unique integral solution $u$ of (1.2) with $u(0) = u_0$. This integral solution for the appropriate $m$-accretive operator in $L^1(\Omega) \times L^1(Q)$ will be the "generalized solution" of (1.1.).

## 2. The Stationary Problem.

We begin with some notation. For $1 \le p \le \infty$ we denote by $L^p(\Omega)$ the usual Lebesgue space, and the conjugate exponent by $p'$, so $\frac{1}{p} + \frac{1}{p'} = 1$. $W^{m,p}(\Omega)$ is the Banach space of those functions whose derivatives up to order $m$ belong to $L^p(\Omega)$, $C_0^\infty(\Omega)$ is the space of infinitely differentiable functions with compact support in $\Omega$, and $W_0^{m,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$. See [1] for information on these Sobolev spaces. Specifically, let $\tau_0 : W^{1,p}(\Omega) \to L^p(\Gamma)$ and $\tau_x : W^{1,p}(\Omega_x) \to L^p(\Gamma_x)$ denote the *trace* maps onto boundary values.

An essential construction is the *continuous direct sum* or *random Banach space* denote by $L^p(\Omega, L^p(\Omega_x))$. We identify $L^p(\Omega, L^p(\Omega_x)) \cong L^p(Q)$ by the Fubini-Tonelli theorems with the duality

$$\int_\Omega \left\{ \int_{\Omega_x} W(x,y)\Phi(x,y)\, dy \right\} dx = \int_Q W\Phi\,, \qquad W \in L^p(Q)\,,\ \Phi \in L^{p'}(Q)\,.$$

Since $W^{1,p}(\Omega_x)$ is continuously imbedded in $L^p(\Omega_x)$, uniformly for $x \in \Omega$, we can identify

$$L^p(\Omega, W^{1,p}(\Omega_x)) = \{W \in L^p(Q) : W(x) \in W^{1,p}(\Omega_x)\,,$$

$$\text{a.e. } x \in \Omega\,, \text{ and } \int_\Omega \|W(x)\|_{W^{1,p}}^p\, dx < \infty\}\,.$$

Similarly we construct $L^p(\Omega, L^p(\Gamma_x))$. We shall assume the trace maps $\tau_x : W^{1,1}(\Omega_x) \to L^1(\Gamma_x)$ are *uniformly bounded* and define the *distributed trace* $\tau(W)$ in $L^1(\Omega, L^1(\Gamma_x))$ for each $W \in L^1(\Omega, W^{1,1}(\Omega_x))$ by $\tau(W)(x,s) = \tau_x(W(x))(s)$, $s \in \Gamma_x$, $x \in \Omega$. Thus $\tau : L^1(\Omega, W^{1,1}(\Omega_x)) \to L^1(\Omega, L^1(\Gamma_x))$ is continuous. Finally, the (constant) imbedding $\lambda : L^1(\Omega) \to L^1(\Omega, L^1(\Gamma_x))$ given by $(\lambda u)(x,s) = u(x)$, $s \in \Gamma_x$, $x \in \Omega$, and its formal dual $\tilde{\lambda} : L^1(\Omega, L^1(\Gamma_x)) \to L^1(\Omega)$ given by $\tilde{\lambda}(U)(x) = \int_{\Gamma_x} U(x,s)\, ds$, $x \in \Omega$, will play an essential role in the porous media system.

The linear Neumann problem

$$(2.1) \qquad -\Delta w = f \ \text{ in } \ L^1(\Omega) \quad , \quad \vec{\nabla} w \cdot \vec{n} = g \ \text{ in } \ L^1(\Gamma)$$

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is an ingredient of each of the boundary-value problems below. By a *solution* we mean a

$$(2.1') \qquad w \in W^{1,1}(\Omega) : \int_\Omega \vec{\nabla} w \cdot \vec{\nabla} \varphi = \int_\Omega f\varphi + \int_\Gamma g\tau_0(\varphi) \ , \qquad \varphi \in W^{1,\infty}(\Omega) \ .$$

By setting $\varphi = 1$ it follows that a necessary condition for the existence of a solution is

$$\int_\Omega f + \int_\Gamma g = 0 \ .$$

It is well-known that this condition is also sufficient for existence of a solution. Solutions are unique up to an additive constant, so there is a unique solution $w$ with $\int_\Omega w = 0$. This solution satisfies

$$\|w\|_{(\Omega)} \leq \text{const.} \ \|f\|_{L^1(\Omega)} + \|g\|_{L^1(\Gamma)} \ .$$

$W^{1,1}(\Omega)$

For a maximal monotone graph $\sigma$ in $\mathbb{R}$ we say $\xi \in \sigma(w)$ in $L^p(\Omega)$ for a real-valued function $w$ on $\Omega$ if $\xi \in L^p(\Omega)$ and $\xi(x) \in \sigma(w(x))$ for a.e. $x \in \Omega$. The following result plays a pivotal role in the following.

**Lemma 1.** *Let $1 \leq p \leq \infty$, $f \in L^p(\Omega)$, $g \in L^p(\Gamma)$ and $w$ be a solution of (2.1). Let $\sigma$ be a maximal monotone graph in $\mathbb{R}$ and $0 \in \sigma(0)$. If $\xi \in \sigma(w)$ in $L^{p'}(\Omega)$ and $\eta \in \sigma(\tau_0 w)$ in $L^{p'}(\Gamma)$, then*

$$\int_\Omega f\xi + \int_\Gamma g\eta \geq 0 \ .$$

< space

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*For a Lipschitz $\sigma$ the result is immediate. For the more general case it follows by the methods of [5]; see [4] for details.*

The *stationary problem* corresponding to (1.1) is the following: for $f \in L^1(\Omega)$ and $F \in L^1(\Omega, L^1(\Omega_x)) = L^1(Q)$ given, find

$$w \in W^{1,1}(\Omega) \ , \quad u \in a(w) \text{ in } L^1(\Omega) \ , \quad v \in b(\tau_0 w) \text{ in } L^1(\Gamma) \ ,$$

$$W \in L^1(\Omega, W^{1,1}(\Omega_x)) \ , \quad U \in \alpha(W) \text{ in } L^1(Q) \ , \quad V \in \beta(\tau W) \text{ in } L^1(\Omega, L^1(\Gamma_x))$$

such that

$$u - \Delta w + \tilde{\lambda}\Big(\gamma(\lambda(u)) - V\Big) = f \qquad \text{in } L^1(\Omega) \tag{2.2.a}$$

$$v + \vec{\nabla} w \cdot \vec{n} = 0 \qquad \text{in } L^1(\Gamma) \tag{2.2.b}$$

$$U - \Delta_y W = F \qquad \text{in } L^1(Q) \tag{2.2.$\alpha$}$$

$$\vec{V} + \vec{\nabla}_y W \cdot \vec{n}_x = \gamma(\lambda(u)) \qquad \text{in } L^1(\Omega, L^1(\Gamma_x)) \ . \tag{2.2.$\beta$}$$

**Proposition 1.** *Assume $\gamma$ is continuous and $|\gamma(r)| \le K|r|$, $r \in \mathbb{R}$. For $j = 1,2$, let $f_j \in L^1(\Omega)$ and $F_j \in L^1(Q)$ be given, and suppose $u_j, v_j, w_j$, $U_j, V_j, W_j$ are corresponding solutions of (2.2). Then*

$$\begin{aligned} \int_\Omega |u_1 - u_2| + \int_\Gamma |v_1 - v_2| + \int_\Omega |\Gamma_x|\,|\gamma(u_1) - \gamma(u_2)| \\ \le \int_\Omega |f_1 - f_2| + \int_\Omega \int_{\Gamma_x} |V_1 - V_2| \ , \end{aligned} \tag{2.3.a}$$

$$\begin{aligned} \int_\Omega \int_{\Omega_x} |U_1 - U_2| + \int_\Omega \int_{\Gamma_x} |V_1 - V_2| \\ \le \int_\Omega \int_{\Omega_x} |F_1 - F_2| + \int_\Omega |\Gamma_x|\,|\gamma(u_1) - \gamma(u_2)| \ , \end{aligned} \tag{2.3.b}$$

*and, hence,*

$$\int_\Omega |u_1 - u_2| + \int_\Gamma |v_1 - v_2| + \int_\Omega \int_{\Omega_x} |U_1 - U_2| \le \int_\Omega |f_2 - f_2| + \int_\Omega \int_{\Omega_x} |F_1 - F_2| . \tag{2.4}$$

**Corollary 1.** *The solutions satisfy*

$$\begin{aligned} \int_\Omega (u_1 - u_2)^+ + \int_\Gamma (v_1 - v_2)^+ + \int_\Omega |\Gamma_x| \big(\gamma(u_1) - \gamma(u_2)\big)^+ \\ \le \int_\Omega (f_1 - f_2)^+ + \int_\Omega \int_{\Gamma_x} (V_2 - V_2)^+ \ , \end{aligned} \tag{2.5.a}$$

$$\begin{aligned} \int_\Omega \int_{\Omega_x} (U_1 - U_2)^+ + \int_\Omega \int_{\Gamma_x} (V_1 - V_2)^+ \\ \le \int_\Omega \int_{\Omega_x} (F_1 - F_2)^+ + \int_\Omega |\Gamma_x| \big(\gamma(u_1) - \gamma(u_2)\big)^+ \ , \end{aligned} \tag{2.5.b}$$

$$(2.6)\qquad \int_\Omega (u_1-u_2)^+ + \int_\Gamma (v_1-v_2)^+ + \int_\Omega\int_{\Omega_x} (U_1-U_2)^+ \le \int_\Omega (f_1-f_2)^+ + \int_\Omega\int_{\Omega_x} (F_1-F_2)^+ .$$

*Proof.* For the two solutions subtract the corresponding equations to obtain (2.2) with $u = u_1 - u_2$, $v = v_1 - v_2$, etc. Note that $w$ is a solution of a linear Neumann problem. Let $\sigma$ be the graph sgn, *i.e.*, $\sigma(r) = 1$ if $r > 0$, $\sigma(r) = -1$ if $r < 0$, and $\sigma(0) = [-1, 1]$. Choose $\xi(x) = \sigma_0(u(x) + w(x))$, $\eta(x) = \sigma_0(v(s) + \tau_0 w(s))$; note that $\xi \in \sigma(u)$, $\xi \in \sigma(w)$ since $a$ is monotone, and $\eta \in \sigma(v)$, $\eta \in \sigma(\tau_0 w)$ since $b$ is monotone. Thus Lemma 1 applies, and we obtain (2.3.a). Similarly, $W(x, \cdot)$ is a solution of a Neumann problem on $\Omega_x$ for a.e. $x \in \Omega$, so the preceding argument at a.e. $x \in \Omega$, followed by an integration over $\Omega$, leads to (2.3.b). Also (2.4) follows by adding (2.3). Finally, the argument above with $\sigma$ chosen to be the graph $\mathrm{sgn}^+$ yields Corollary 1.

The inequality (2.4) is the fundamental estimate in $L^1(\Omega) \times L^1(Q)$ on a solution of the system (2.2). It shows that the dynamics of (1.1) corresponds to a semigroup of contractions in this space. Moreover, (2.3) displays explicitly the dependence of the *components* of (2.2) on the exchange or coupling terms in $L^1(\Omega, L^1(\Gamma_x))$.

**Corollary 2.** *Suppose $f \in L^1(\Omega)$ and $F \in L^1(Q)$. For $j = 1, 2$, let $V_j \in L^1(\Omega, L^1(\Gamma_x))$ be given and suppose $u_j, v_j, w_j$ are corresponding solutions of (2.2.a), (2.2.b). Then*

$$(2.7.a)\qquad \int_\Omega |u_1 - u_2| + \int_\Omega |\Gamma_x|\, |\gamma(u_1) - \gamma(u_2)| \le \int_\Omega\int_{\Gamma_x} |V_1 - V_2| .$$

*Similarly, let $u_j$ be given for $j = 1, 2$ and suppose $U_j, V_j, W_j$ are corresponding solutions of (2.2.$\alpha$), (2.2.$\beta$). Then*

$$(2.7.b)\qquad \int_\Omega\int_{\Omega_x} |U_1 - U_2| + \int_\Omega\int_{\Gamma_x} |V_1 - V_2| \le \int_\Omega |\Gamma_x|\, |\gamma(u_1) - \gamma(u_2)| .$$

**Theorem 1.** *Assume the following of the maximal monotone graphs $a, b, \alpha, \beta, \gamma$:*

$$Rg(a + b) = \mathbb{R} \quad \text{and} \quad b[\operatorname{dom} a] \ni 0 ;$$

$$Rg(\alpha + \beta) = \mathbb{R} \quad \text{and} \quad \beta[\operatorname{dom} \alpha] \supset Rg\, \gamma \circ a ;$$

*$\gamma$ is Lipschitz on $Rg\,a$: $|\gamma(r_1)-\gamma(r_2)|\le K|r_1-r_2|$, $r_1,r_2\in Rg\,a$. Then for each $f\in L^1(\Omega)$, $F\in L^1(Q)$ there is a solution of (2.2).*

*Proof.* Let $\overline{V}\in L^1(\Omega,L^1(\Gamma_x))$ be given. From [4] it follows there exists a solution $u,v,w$ of (2.2.a) with $V$ replaced by $\overline{V}$, (2.2.b), $u\in a(w)$ in $L^1(\Omega)$, and $v\in b(\tau_0 w)$ in $L^1(\Gamma)$. Fix $x\in\Omega$ and consider the boundary-value problem

(2.8.a) $$W(x)\in W^{1,1}(\Omega_x)\ ,\ U(x)\in\alpha\big(W(x)\big)\text{ in }L^1(\Omega_x)\ ,$$
$$V(x)\in\beta\big(\tau_x W(x)\big)\text{ in }L^1(\Gamma_x)$$

(2.8.b) $$U(x)-\Delta_y W(x)=F(x)\ \text{ in }\ L^1(\Omega_x)\ ,$$

(2.8.c) $$V(x)+\vec{\nabla}_y W(x)\cdot\vec{n}_x=\gamma\big((\lambda u)(x)\big)$$

By use of $[r,s]\in\alpha$ with $\beta(r)\ni\gamma(\lambda u(x))$ we construct the graphs $\tilde{\alpha}(t)=\alpha(t+r)-s$ and $\tilde{\beta}(t)=\beta(t+r)-\gamma(\lambda u(x))$ for which (2.8) is equivalent to

$$\widetilde{W}\in W^{1,1}(\Omega_x)\quad,\quad\widetilde{U}\in\tilde{\alpha}(\widetilde{W})\quad,\quad\widetilde{V}\in\tilde{\beta}(\tau_x\widetilde{W})$$

$$\widetilde{U}-\Delta_y\widetilde{W}=F(x)-s\ \text{ in }\ L^1(\Omega_x)$$

$$\widetilde{V}+\vec{\nabla}_y\widetilde{W}\cdot\vec{n}_x=0\ \text{ in }\ L^1(\Gamma_x)$$

for the functions $\widetilde{U}=U(x)-s$, $\widetilde{V}=V(x)-\gamma(\lambda u(x))$, $\widetilde{W}=W(x)-r$, and $0\in\tilde{\alpha}(0)$, $0\in\tilde{\beta}(0)$. Thus (2.8) has a solution $U(x),V(x),W(x)$ by [4] for a.e. $x\in\Omega$, and from the estimates above this family of solutions is a solution $U,V,W$ of (2.2.$\alpha$), (2.2.$\beta$). Set $V=T(\overline{V})$; this defines a self-map $T$ on $L^1(\Omega,L^1(\Gamma_x))$, and from the estimates (2.3) it follows that $T$ is non-expansive. Moreover, if $\overline{V}_1,\overline{V}_2\in L^1(\Omega,L^1(\Gamma_x))$ and the corresponding solutions of (2.2) with $V$ replaced by $\overline{V}_j$ in (2.2.a) are denoted by $u_j,v_j,w_j,U_j,V_j,W_j$ for $j=1,2$, then (2.7.a) with $V_j=\overline{V}_j$ and (2.7.b) hold. If $M=\sup\{|\Gamma_x|:x\in\Omega\}$, then we have from the Lipschitz condition on $\gamma$

$$\Big(\frac{1}{KM}+1\Big)\int_\Omega|\Gamma_x|\,|\gamma(u_1)-\gamma(u_2)|\le\int_\Omega|u_1-u_2|+\int_\Omega|\Gamma_x|\,|\gamma(u_1)-\gamma(u_2)|\,.$$

Thus with (2.7) we obtain

$$\int_\Omega\int_{\Gamma_x}|T(\overline{V}_1)-T(\overline{V}_2)|\le\Big(1+\frac{1}{KM}\Big)^{-1}\int_\Omega\int_{\Gamma_x}|\overline{V}_1-\overline{V}_2|$$

so $T$ is a strict contraction. The fixed point $V = T(V)$ in $L^1(\Omega, L^1(\Omega_x))$ yields the solution of (2.2).

Define a (possibly multivalued) operator $A$ in $L^1(\Omega) \times L^1(Q)$ as follows: $[f, F] \in A[u, U]$ if there exist $w \in W^{1,1}(\Omega)$, $W \in L^1(\Omega, W^{1,1}(\Omega_x))$ and $u, v$ for which

$$
\begin{aligned}
&-\Delta w + \tilde{\lambda}\big(\gamma(\lambda u) - V\big) = f \,, \quad u \in a(w) && \text{in } L^1(\Omega) \,, \\
&v + \vec{\nabla} w \cdot \vec{n} = 0 \,, \quad v \in \beta(\tau_0 w) && \text{in } L^1(\Gamma) \,, \\
&-\Delta_y W = F \,, \quad U \in \alpha(W) && \text{in } L^1(Q) \,, \\
&V + \vec{\nabla}_y W \cdot \vec{n}_x = \gamma(\lambda u) \,, \quad V \in \beta(\tau W) && \text{in } L^1(\Omega, L^1(\Gamma_x)) \,.
\end{aligned}
$$

Thus (2.2) is equivalent to $(I + A)[u, U] \ni [f, F]$. From (2.4) and Theorem 1 it follows that $(I + A)^{-1}$ is a contraction on $L^1(\Omega) \times L^1(Q)$. By rescaling the five graphs it follows easily that the same holds for $(I + \varepsilon A)^{-1}$ for every $\varepsilon > 0$, so $A$ is *m-accretive* on $L^1(\Omega) \times L^1(Q)$.

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# A PROCEDURE FOR CALCULATING VORTICITY BOUNDARY CONDITIONS IN THE STREAM-FUNCTION–VORTICITY METHOD

R. J. MACKINNON[2], G. F. CAREY[1] AND P. MURRAY[1]
[1] *The University of Texas at Austin, Austin, Tx 78712, U.S.A.*
[2] *EG&G Idaho, Inc., Idaho Falls, ID. 83415, U.S.A.*

## SUMMARY

A new superconvergent projection formula for determining vorticity boundary data in the stream-function–vorticity method is constructed.

## INTRODUCTION

The stream-function–vorticity $(\psi, \omega)$ formulation is a standard approach for numerical treatment of 2D viscous flows. In this procedure the problem reduces to solution of a coupled pair of partial differential equations – the vorticity transport equation and stream-function equation. These equations can be discretized and iteratively decoupled, and then solved for iterates approximating $\psi$ and $\omega$. A well-known difficulty in this algorithm is the problem of specifying vorticity boundary data as essential data for the vorticity transport equation. It is standard practice on rectangular finite-difference grids to use one-sided finite differences of the stream-function iterate to compute an approximation to the velocity and thereby the bounaary vorticity.[1] A similar procedure can be used in finite-element methods but does not fit naturally in this framework. Here we present an alternative approach based on superconvergent flux ideas that applies to both straight and curved boundaries and can be used for either finite-element or finite-difference computations.

## FORMULATION

Recall that in the stream-function–vorticity method the stream function $\psi$ satisfies the Poisson equation

$$-\Delta\psi = \omega \text{ in } \Omega \tag{1}$$

Here $\omega$ is the vorticity determined from the vorticity transport equation (e.g. for steady Stokes flow $\Delta\omega = f$). The stream-function–vorticity equations are frequently iteratively decoupled in the numerical solution scheme. The objective here is to construct a proceudre that exploits superconvergence ideas to develop a post-processing formula from (1) for approximating the vorticity on the boundary. This can then be used as data for the vorticity transport equation.

We introduce the familiar Green–Gauss formula for the Laplacian operator

$$\int_\Omega (-\Delta u)v \, dx \, dy = \int_\Omega \nabla u \cdot \nabla v \, dx \, dy - \int_{\partial\Omega} v \frac{\partial u}{\partial n} ds \tag{2}$$

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where $u$, $v$ are arbitrary admissible functions. In particular, let us select $u = \psi$ satisfying (1) so that

$$\int_{\Omega} \omega v \, \mathrm{d}x \, \mathrm{d}y = \int_{\Omega} \nabla\psi \cdot \nabla v \, \mathrm{d}x \, \mathrm{d}y - \int_{\partial\Omega} v \frac{\partial \psi}{\partial n} \, \mathrm{d}s \tag{3}$$

Now (3) is an identity satisfied by the solution $(\psi, \omega)$ for arbitrary admissible $v$. In previous superconvergence studies, a similar construction has been developed and the approximate solution introduced to obtain superconvergent boundary flux approximations (i.e. for $\partial\psi/\partial n$) (e.g. see Wheeler[2], Carey[3]). In the present case we instead use the known boundary data $\partial\psi/\partial n = u_s$, where $s$ is the tangential direction. Then for known $\psi$ and $\partial\psi/\partial n$ in (3) we have a projection formula for vorticity in $\Omega$. Now set the approximate solution $\psi_h$ for $\psi$ on the discretized domain $\Omega_h$ with $v = \phi_i$, the piecewise-polynomial Lagrange basis function associated with node $i$ on the boundary, to get the approximate projection (for $\omega^* \approx \omega$)

$$\int_{\Omega_h} \omega^* \phi_i \, \mathrm{d}x \, \mathrm{d}y = \int_{\Omega_h} \nabla\psi_h \cdot \nabla\phi_i \, \mathrm{d}x \, \mathrm{d}y - \int_{\partial\Omega_h} u_s \phi_i \, \mathrm{d}s \tag{4}$$

Now as $i$ traverses the boundary nodes the integral on the left involves only the strip of elements adjacent to the boundary. Furthermore, since $\phi_i(x_j, y_j) = \delta_{ij}$, using a Lobatto (node point) quadrature on the left simplifies the expression to yield an explicit superconvergent extraction formula approximating the vorticity at boundary node $i$ within quadrature accuracy as

$$Q_i \omega_i^* = \int_{\Omega_h} \nabla\psi_h \cdot \nabla\phi_i \, \mathrm{d}x \, \mathrm{d}y - \int_{\partial\Omega_h} u_s \phi_i \, \mathrm{d}s \tag{5}$$

where $Q_i$ corresponds to the accumulated quadrature weight at node $i$ from the adjacent elements.

*Remarks*

1. For a rectilinear boundary and bilinear elements the extraction formula (5) is equivalent to a one-sided second-order difference approximation.[4] For higher-degree elements, more general boundary shapes and irregular grids, (5) is still applicable. The scheme has been applied in finite-element calculations for viscous flow applications with straight and curved boundary geometries as well as moving surfaces (Murray[5]).
2. The scheme (5) can be applied with finite-difference methods by formally introducing the nodal interpolant of the finite-difference solution as $\psi_h$.

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